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Characterizations of Amenable Lau Algebras

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Abstract. In this paper, among the other things, we establish a characterization of a left amenable Lau algebra (which includes the group algebra and the Fourier algebra of a locally compact group and quantum group algebras, or more generally the predual algebra of a Hopf von Neumann algebra) in terms of the Hahn–Banach property. We prove a variety of characterizations of left amenable Lau algebras. We give sufficient conditions and some necessary conditions for a Lau algebra to have a topological left invariant mean.

1. Introduction

There are a lot of results in abstract harmonic analysis on amenability of a Banach algebra. The question of amenability for a Banach algebra was first studied by B. E. Johnson in 1972. Various notions of amenability have been introduced in the recent years. The most recent contributions, to our knowledge, are papers by Bade, Curtis and Dales [2], and by Curtis and Loy [6]. Most notable are the notions of weak amenability, operator amenability (i.e. the notion of amenability in the category of operator spaces), n-weak amenability, permanently weak amenability and Connes amenability. For terminologies regarding amenability of Banach algebras, the reader is referred to [7, 17, 19, 24].

The study of left amenability of F-algebras was initiated by Lau [14] and pursued by Lau and Zhang [16]. Amenability and inner amenability of F-algebras is studied in [5, 8, 21]. The concept of left amenability for a F-algebra has been extensively extended for an arbitrary Banach

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algebra by introducing the notion of φ-amenability, see [12, 13]. Character amenability has also been studied recently by by Hu, Monfared and Traynor in [11]. Both of these concepts generalize the earlier concept of left amenability for F-algebras introduced by Lau in [14], for more information see [18].

Let A be a F-algebra. The purpose of this paper is to study a notion of left amenability for an arbitrary F-algebra. We obtain the necessary and sufficient conditions for F-algebra A to have a topological left invariant mean. We obtain some equivalent conditions for left amenability of A in terms of the Hahn–Banach property. We then apply these results to certain F-algebras on a locally compact group G to give characterizations for amenability of G.

2. Preliminaries and Notations

Throughout this paper, A will denote a F-algebra; that is a complex Banach algebra which is the unique predual of a W^*-algebra M and the identity element of M is a multiplicative linear functional on A. As pointed out in [14], such an M is not necessarily unique as a dual von-Neumann algebra of A, although it is plainly unique as the dual space of A. The subject of this class of Banach algebras originated with a paper published in 1983 by Lau [14] in which he referred to them as "F-algebras". Later on, in his useful monograph Pier [22] introduced the name "Lau algebra", see also [1].

The class of Lau algebras includes many classical Banach algebras related to a locally compact group or a hypergroup, for more information see [4, 9]. It also includes the predual algebras of Hopf von Neumann algebras, in particular the class of quantum group algebras $L^1(G)$.

The notion of left amenability for Lau algebras was introduced by Lau [14]. In the same paper he extended several characterizations of amenable locally compact groups to left amenable Lau algebras; see also Ghahramani and Lau [10]. Let us recall from [14] that any commutative Lau algebra is left amenable; in particular, $A(G)$ and $B(G)$ of locally compact groups G are always left amenable. Also $L^1(G)$ (resp. $M(G)$) is left amenable if and only if G is amenable; see Pier [22] for details on amenable locally compact groups. It is shown that a Lau algebra A is left amenable if and only if the semigroup S of normal positive functionals of norm 1 on A^* has a fixed point property on a compact Hausdorff space [8].

For a locally convex space X, the dual space of X is denoted by X^*. The action of $f \in X^*$ at $x \in X$ is denoted either by $f(x)$ or by $\langle f, x\rangle$. The second dual A^{**} of A equipped with the first Arens multiplication is a Lau algebra [14], where is defined by the equations

$$\langle FG, f\rangle = \langle F, Gf\rangle, \qquad \langle Gf, a\rangle = \langle G, fa\rangle, \qquad \langle fa, b\rangle = \langle f, ab\rangle$$

for all $F, G \in A^{**}$, $f \in A^*$ and $a, b \in A^*$, see also [3, 15].

The identity of A^* will be denoted by 1. Also $P(A)$ will denote the cone of all positive functionals in A and $P^1(A)$ will denote the set of all a in $P(A)$ such that $\langle 1, a\rangle = 1$. A is called left amenable if there exists a bounded linear functional m on A^* satisfying $\langle m, 1\rangle = \|m\| = 1$ and $\langle m, fa\rangle = \langle m, f\rangle$ for all $f \in A^*$ and $a \in P^1(A)$ [8]. The functional m is called a topological left invariant mean on A^*. This concept generalizes the concept of amenability for locally compact groups. The convex set of topological left invariant means on A^* is denoted by $TLIM(A)$. It is known that a Lau algebra A is left amenable if for any two-sided Banach A-module X such that $a.x = \langle 1, a\rangle x$ for all $a \in A$ and $x \in X$, every bounded derivation from A into X^* is inner [14].

3. Main Results

Our starting point of this section is the following Theorem. Note that this theorem is in fact Proposition 15.5 of [22] which was proved for groups. However, the proof is completely different.

Theorem 3.1. *Let A be a Lau algebra. Then the following conditions are equivalent:*

(i) *A is left amenable;*

(ii) *For all $n \in \mathbb{N}$ and all $a_1, \ldots, a_n \in A$,*

$$\begin{aligned}&\inf\left\{\sup\{\|a_i a\|; i \in \{1, \ldots, n\}\}; a \in P^1(A)\right\}\\ &\quad \leq \sup\{|\langle 1, a_i\rangle|; i \in \{1, \ldots, n\}\}.\end{aligned}$$

Proof. (i) implies (ii). If A is left amenable, then there exist a net $\{a_\alpha\}$ in $P^1(A)$ such that $\|aa_\alpha - a_\alpha\| \to 0$ for all $a \in P^1(A)$ [14]. Let $a_1, \ldots, a_n \in A$ and write

$$a_i = \sum_{k=1}^{n_i} \lambda_k^i a_k^i, \quad \text{where} \quad \lambda_k^i \in \mathbb{C}, a_k^i \in P^1(A) \quad \text{and} \quad i = \{1, \ldots, n\}.$$

Then

$$|\langle 1, a_i\rangle| = \left|\sum_{k=1}^{n_i} \lambda_k^i\right| \quad \text{for all} \quad i \in \{1, \ldots, n\}.$$

Choose $\epsilon > 0$ and put $\delta \max\left\{n_i\left|\lambda_k^i\right|; 1 \leq k \leq n_i, 1 \leq i \leq n\right\} = \epsilon$. There exists α_0 such that $\left\|a_k^i a_{\alpha_0} - a_{\alpha_0}\right\| < \delta$ for all $i \in \{1, \ldots, n\}$ and $1 \leq k \leq n_i$. Then for $i = \{1, \ldots, n\}$,

$$\begin{aligned}
\|a_i a_{\alpha_0}\| &\leq \left\|\sum_{k=1}^{n_i} \lambda_k^i a_k^i a_{\alpha_0} - \sum_{k=1}^{n_i} \lambda_k^i a_{\alpha_0}\right\| + \left\|\sum_{k=1}^{n_i} \lambda_k^i a_{\alpha_0}\right\| \\
&\leq \sum_{k=1}^{n_i} \left|\lambda_k^i\right| \left\|a_k^i a_{\alpha_0} - a_{\alpha_0}\right\| + \left|\sum_{k=1}^{n_i} \lambda_k^i\right| \\
&\leq \delta \sum_{k=1}^{n_i} \left|\lambda_k^i\right| + |\langle 1, a_i\rangle| \\
&< \epsilon + |\langle 1, a_i\rangle| .
\end{aligned}$$

Hence $\sup\{\|a_i a_{\alpha_0}\|; i \in \{1, \dots, n\}\} \leq \epsilon + \sup\{|\langle 1, a_i\rangle|; i \in \{1, \dots, n\}\}$. This shows that

$$\begin{aligned}
&\inf\left\{\sup\{\|a_i a\|; i \in \{1, \dots, n\}\}; a \in P^1(A)\right\} \\
&\quad \leq \sup\{|\langle 1, a_i\rangle|; i \in \{1, \dots, n\}\}.
\end{aligned}$$

Now let (ii) hold. It is known that the set $\mathcal{M}$ consisting of the means on A^* is weak*-compact in A^{**} [7]. For each finite subset F in $P^1(A)$ and each $\epsilon > 0$ we associated the subset

$$\mathcal{A}(F, \epsilon) = \overline{\{m \in \mathcal{M}; \|am - m\| < \epsilon \ \text{ for all } \ a \in F\}},$$

where closure is taken in the weak*-topology. These sets are weak*-closed subsets of A^{**} and we want to show that

$$\bigcap\left\{\mathcal{A}(F, \epsilon); F \subseteq P^1(A) \quad \text{is finite} \ \text{ and } \ \epsilon > 0\right\}$$

is nonempty. Let $a \in P^1(A)$ be fixed. Let $F = \{a_1, \dots, a_n\} \subseteq P^1(A)$ and $\epsilon > 0$ be given. By hypothesis, there exists $b_1 \in P^1(A)$ such that $\|(a_1 a - a) b_1\| < \epsilon$. Since $\langle 1, (a_2 a - a) b_1\rangle = 0$, again by hypotesis, there exists $b_2 \in P^1(A)$ such that $\|(a_2 a - a) b_1 b_2\| < \epsilon$. Proceeding inductively, we see that there exist $b_n \in P^1(A)$ such that $\|(a_n a - a) b_1 b_2 \cdots b_n\| < \epsilon$. Put $b_{(F,\epsilon)} = b_1 b_2 \cdots b_n$, then $\left\|a_i a b_{(F,\epsilon)} - a b_{(F,\epsilon)}\right\| < \epsilon$ for all $i \in \{1, \dots, n\}$. This shows that $a b_{(F,\epsilon)} \in \mathcal{A}(F, \epsilon)$ and so $\mathcal{A}(F, \epsilon) \neq \emptyset$. Now, if $F_1, \dots, F_m$ are finite subsets of $P^1(A)$ and $\epsilon_1, \dots, \epsilon_m \in \mathbb{R}^+$, let $F = \bigcup\{F_i, 1 \leq i \leq m\}$ and $\epsilon = \min\{\epsilon_1, \dots, \epsilon_m\}$. Therefore

$$\emptyset \neq \mathcal{A}(F, \epsilon) \subseteq \bigcap\{\mathcal{A}(F_i, \epsilon_i); 1 \leq i \leq m\}.$$

Since $\mathcal{M}$ is weak* compact, we must have

$$\bigcap\left\{\mathcal{A}(F, \epsilon); F \subseteq P^1(A) \quad \text{is finite and} \ \ \epsilon > 0\right\} \neq \emptyset.$$

Any mean m in this latter intersection is a topological left invariant mean. Indeed, suppose m is in this latter intersection. Choose $f \in A^*$, $a \in P^1(A)$ and $\epsilon > 0$. Therefore $m \in A(\{a\}, \epsilon)$. Choose $n \in M$ such that $|\langle m,f\rangle - \langle n,f\rangle| < \epsilon$, $|\langle m,fa\rangle - \langle n,fa\rangle| < \epsilon$ and $\|an-n\| < \epsilon$. We have

$$\begin{aligned}|\langle m,f\rangle - \langle m,fa\rangle| &\leq |\langle m,f\rangle - \langle n,f\rangle| + |\langle n,f\rangle - \langle n,fa\rangle| \\ &\quad + |\langle n,fa\rangle - \langle m,fa\rangle| \\ &\leq \epsilon + \|an-n\|\,\|f\| + \epsilon \\ &\leq (2+\|f\|)\,\epsilon.\end{aligned}$$

Since ϵ was arbitrary, we see that $\langle m,f\rangle = \langle m,fa\rangle$. This completes our proof. □

Let S be a semigroup and let $\mathcal{B}(S)$ be the Banach space of all bounded functions $f : S \to \mathbb{C}$ with the sup norm. A linear functional n on $\mathcal{B}(S)$ is a mean if n is positive and of norm one. If for each $f \in \mathcal{B}(S)$ and $s \in S$, $n({}_sf) = n(f)$, where ${}_sf(t) = f(st)$ for each s,t in S, then n is a left invariant mean on $\mathcal{B}(S)$. The set of all left invariant means on $\mathcal{B}(S)$ is denoted by $LIM(S)$. S is said to be amenable if $LIM(S) \neq \emptyset$.

Let A be a Lau algebra. We denote by $S(A)$ the family of all nonempty finite subsets of $P^1(A)$. $S(A)$ becomes a semigroup if we define multiplication by set product, i.e. $BB' = \{bb'; b \in B, b' \in B'\}$ for each B, B' in $S(A)$.

Proposition 3.2. *Let A be a Lau algebra. If $S(A)$ is amenable, then A is left amenable.*

Proof. For each f in A^*, define $\overline{f}$ in $\mathcal{B}(S(A))$ by

$$\overline{f}(B) = \frac{1}{k}\sum_{i=1}^{k}\langle f, b_i\rangle$$

for each $B = \{b_1, \ldots, b_k\} \in S(A)$. By assumption, there is a left invariant mean n on $\mathcal{B}(S(A))$. Define m in A^{**} by $\langle m,f\rangle = n\left(\overline{f}\right)$ for each $f \in A^*$. It is obvious that m is a mean. To see that m is invariant, let $f \in A^*$ and $a \in P^1(A)$. For $B \in S(A)$,

$$\begin{aligned}\overline{fa}(B) &= \frac{1}{k}\sum_{i=1}^{k}\langle fa, b_i\rangle \\ &= \frac{1}{k}\sum_{i=1}^{k}\langle f, aa_i\rangle \\ &= \overline{f}(\{a\}B) \\ &=_{\{a\}}\overline{f}(B).\end{aligned}$$

This shows that $\overline{fa} =_{\{a\}} \overline{f}$. Hence

$$\begin{aligned}\langle m, fa\rangle &= n\left(\overline{fa}\right)\\ &= n\left({}_{\{a\}}\overline{f}\right)\\ &= n\left(\overline{f}\right)\\ &= \langle m, f\rangle,\end{aligned}$$

and thus A is left amenable. □

The following example showing that converse of Proposition 3.2 is not true.

Example 3.3. Let $G = \mathbb{Z}_2 * \mathbb{Z}_2$, the free product of $\mathbb{Z}_2$ with $\mathbb{Z}_2$. Let $A := L^1(G)$ be the group algebra of G. Then $S(A)$ is not amenable.

Proof. It is well known that G is amenable. B. Johnson proved that G is amenable if and only if the algebra $L^1(G)$ is amenable. Therefore $L^1(G)$ is left amenable [14].

Note that G is isomorphic to the discrete group generated by a and b where $a^2 = b^2 = e$. We show that $S(A)$ is not amenable by producing disjoint right ideals in $S(A)$. If I and J are the right ideals in $S(A)$ generated by $\{\delta_a, \delta_{ab}\} \in S(A)$ and $\{\delta_b, \delta_{ba}\} \in S(A)$, respectively, then $I \cap J = \emptyset$. To see this consider the following: each element of g in G has a unique representation as a reduced word $g = a^{\epsilon_1} b^{\epsilon_2} \cdots a^{\epsilon_{n-1}} b^{\epsilon_n}$, where $\epsilon_i = 1$ for $2 \leq i \leq n-1$ and $\{\epsilon_1, \epsilon_n\} \subseteq \{0, 1\}$. Define $s(g) = \sum_{i=1}^{n} \epsilon_i$, and for each B in $S(A)$ define $s(B) = \max\{s(g); \delta_g \in B\}$. If $B \in I$, and $\delta_g \in B$ such that $s(g) = s(B)$ then one can easily see that in reduced word form of g, $\epsilon_1 = 1$. If $C \in J$, and g is chosen similarly, $\epsilon_1 = 0$ in reduced word form of g. Thus $I \cap J = \emptyset$. Since I and J are right ideals of $S(A)$, the proposition follows. □

Let X be a normed space. An action of A on X is a bilinear mapping $T : A \times X \to X$ denoted by $(a, x) \mapsto T_a(x)$ such that $T_{ab} = T_b T_a$ for any $a, b \in A$. We say that subspace Y of X is A-invariant under the action $A \times X \to X$ if $T_a(Y) \subseteq Y$ for any $a \in A$.

Let A be a Lau algebra. A is said to have the Hahn-Banach extension property with respect to A^* if given any separately continuous action T of A on a normed space X, an A-invariant subspace Y of X, a continuous seminorm p on X and a linear functional φ on Y such that

$$p(T_a(x)) \leq p(x), |\langle \varphi, y\rangle| \leq p(y) \quad \text{and} \quad \langle \varphi, T_a(y)\rangle = \langle \varphi, y\rangle,$$

for every $a \in P^1(A)$, $x \in X$ and $y \in Y$. Then φ must have an extension to a linear ϕ on X with $|\langle \phi, x\rangle| \leq p(x)$ and $\langle \phi, T_a(x)\rangle = \langle \phi, x\rangle$ for all $x \in X$ and $a \in P^1(A)$.

We establish a criterion that ensures the existence of topological left invariant means via the Hahn-Banach theorem is, a definitely nonconstructive procedure.

Theorem 3.4. *Let A be a Lau algebra. Then the following conditions are equivalent:*

(i) *A is left amenable;*

(ii) *A has the Hahn-Banach extension property with respect to A^*.*

Proof. Let m be a topological left invariant mean on A^*. Let T, X, Y, p and φ be such that the conditions of the Hahn-Banach extension property are satisfied. By [23, Theorem 3.2], φ extends to a linear functional ψ on X that also satifies $|\psi(x)| \leq p(x)$ for all $x \in X$. If $\{x_n\}$ is a sequence in X such that $x_n \to 0$, then $|\langle \psi, x_n \rangle| \leq p(x_n) \to 0$. Hence $\psi \in X^*$. Let x be a fixed element in X. The mapping $\psi_x : a \mapsto \langle \psi, T_a(x) \rangle$ belongs to A^*. Define ϕ on X by $\langle \phi, x \rangle = \langle m, \psi_x \rangle$ for every $x \in X$. Since $\psi_{\alpha x + \beta y} = \alpha \psi_x + \beta \psi_y$, ϕ is linear. On the other hand,

$$\begin{aligned} \langle \psi_y, a \rangle &= \langle \psi, T_a(y) \rangle \\ &= \langle \varphi, T_a(y) \rangle \\ &= \langle \varphi, y \rangle, \end{aligned}$$

for each $y \in Y$ and $a \in P^1(A)$. Thus

$$\begin{aligned} \langle \phi, y \rangle &= \langle m, \psi_y \rangle \\ &= \langle m, 1 \rangle \langle \varphi, y \rangle \\ &= \langle \varphi, y \rangle, \end{aligned}$$

and so ϕ extends φ. For every $a, b \in A$ and $x \in X$, we have

$$\begin{aligned} \left\langle \psi_{T_a(x)}, b \right\rangle &= \langle \psi, T_b(T_a(x)) \rangle \\ &= \langle \psi, T_{ab}(x) \rangle \\ &= \langle \psi_x a, b \rangle. \end{aligned}$$

Thus

$$\begin{aligned} \langle \phi, T_a(x) \rangle &= \langle m, \psi_{T_a}(x) \rangle \\ &= \langle m, \psi_x a \rangle \\ &= \langle m, \psi_x \rangle \\ &= \langle \phi, x \rangle \end{aligned}$$

for all $a \in P^1(A)$ and $x \in X$. Finally, as m is a mean $|\langle m, \psi_x \rangle| \leq \|\psi_x\|$. Since $P^1(A)$ is weak*-dense in $\mathcal{M}$ [14], there is a net $\{a_\alpha\}$ in $P^1(A)$ such that $\{a_\alpha\}$ converges to m in the weak* topology. It follows that

$$|\langle \phi, x \rangle| = |\langle m, \psi_x \rangle|$$

$$
\begin{aligned}
&= \lim_{\alpha} |\langle a_\alpha, \psi_x\rangle| \\
&= \lim_{\alpha} |\langle \psi, T_{a_\alpha}(x)\rangle| \\
&\le \sup\{p(T_{a_\alpha}(x)); \alpha \in I\} \\
&\le \sup\{p(x); \alpha \in I\} \\
&= p(x).
\end{aligned}
$$

Therefore A has the Hahn-Banach extension property.

To prove the converse, assume that A has the Hahn-Banach extension property with respect to A^*. Let $X = A^*$, $Y = \{\alpha 1; \alpha \in \mathbb{C}\}$ and let $T_a(f) = fa$ for all $a \in A$ and $f \in A^*$. It is clear that $T : A \times A^* \to A^*$ is bilinear and separately continuous. Since

$$
\begin{aligned}
T_a(T_b(f)) &= T_a(fb) \\
&= fba \\
&= T_{ba}(f),
\end{aligned}
$$

T is an antihomomorphism. Define $\varphi \in Y^*$ by $\langle \varphi, \alpha 1\rangle = \alpha$. Define a seminorm p on A^* by $p(f) = \|f\|$. Clearly the definition of the Hahn-Banach extension property has also been satisfied. Therefore φ has an extension $m \in A^{**}$ such that $|\langle m, f\rangle| \le \|f\|$ and $\langle m, T_a(f)\rangle = \langle m, f\rangle$ for all $f \in A^*$ and $a \in P^1(A)$. This shows that $\|m\| \le 1$. Since $\langle m, 1\rangle = \langle \varphi, 1\rangle = 1$, m is a topological left invariant mean on A^*, and so A is left amenable. □

Since the Fourier algebra $A(G)$ of a locally compact group G is always left amenable [22], $A(G)$ has the Hahn-Banach extension property.

Let G be a locally compact group with a fixed left Haar measure λ, and let $L^p(G)$ $(1 \le p \le \infty)$ have the usual meanings [7]. Let $WAP\left(L^1(G)\right)$ denote the space of weakly almost periodic functions on G i.e. all $f \in L^\infty(G)$ such that $\{L_x f; x \in G\}$ is relatively compact in the weak topology of $L^\infty(G)$. It is known that $f \in WAP\left(L^1(G)\right)$ if and only if $\left\{f\varphi; \varphi \in P^1(G)\right\}$ is relatively weakly compact if and only if $\left\{f\varphi; \varphi \in L^1(G), \|\varphi\|_1 \le 1\right\}$ is relatively weakly compact [7]. It is not difficult to see that $WAP\left(L^1(G)\right) \subseteq C_b(G)$. Recall that an application of the Ryll-Nardzewski fixed point theorem shows that $WAP\left(L^1(G)\right)$ has a unique invariant mean m. Obviously m is a topological left invariant mean on $WAP\left(L^1(G)\right)$ [22].

Let A be a Lau algebra. A functional $f \in A^*$ is weakly almost periodic if $\{fa; \|a\| \le 1\}$ is relatively weakly compact in A^*. The set of all weakly almost periodic functionals on A is denoted by $WAP(A)$; it is a closed subspace of A^*.

Theorem 3.5. *Let A be a Lau algebra for which $WAP(A)$ has a topological left invariant mean. Let T be a separately weakly continuous action of A on a Banach space X such that $\|T_a\| \leq 1$ for each $a \in P^1(A)$. Suppose that $O_x = \{T_a(x); a \in P^1(A)\}$ is relatively weakly compact for each $x \in X$. Let Y be a closed subspace of X invariant under T and $\varphi \in Y^*$ be such that $\langle \varphi, T_a(y) \rangle = \langle \varphi, y \rangle$ for every $a \in P^1(A)$ and $y \in Y$. Then φ has an extension $\phi \in X^*$ such that $\|\varphi\| = \|\phi\|$ and $\langle \phi, T_a(x) \rangle = \langle \phi, x \rangle$ for every $x \in X$ and $a \in P^1(A)$.*

Proof. By the Hahn-Banach theorem φ extends to some $\psi \in X^*$ with $\|\psi\| = \|\varphi\|$ [23]. Let $x \in X$. The mapping $\psi_x : a \mapsto \psi(T_a(x))$ belongs to A^*. Since O_x is relatively weakly compact, $\psi_x \in WAP(A)$. Indeed, we can define a transformation L of X into A^* by $L(x) = \psi_x$. Then L is obviously linear and

$$\begin{aligned} |\langle L(x), a \rangle| &= |\langle \psi_x, a \rangle| \\ &= |\langle \psi, T_a(x) \rangle| \\ &\leq \|\psi\| \|x\| \|a\| . \end{aligned}$$

Thus $\|L(x)\| \leq \|\psi\| \|x\|$, and so L is a bounded linear transformation. Therefore L is also continuous when both X and A^* carry their weak topologies [23]. In particular the hypothesis about O_x gives that $L(O_x)$ is relatively weakly compact in A^*. On the other hand,

$$\begin{aligned} \left\langle \psi_{T_a(x)}, b \right\rangle &= \langle \psi, T_b(T_a(x)) \rangle \\ &= \langle \psi, T_{ab}(x) \rangle \\ &= \langle \psi_x a, b \rangle , \end{aligned}$$

for every $a, b \in A$, and so $\psi_{T_a(x)} = \psi_x a$. Therefore

$$\begin{aligned} L(O_x) &= \left\{ L(T_a(x)); a \in P^1(A) \right\} \\ &= \left\{ \psi_{T_a(x)}; a \in P^1(A) \right\} \\ &= \left\{ \psi_x a; a \in P^1(A) \right\} . \end{aligned}$$

This shows that $\{\psi_x a; a \in P^1(A)\}$ is relatively weakly compact. By Krein Smulian's Theorem, the closed, convex, circled hull of $\{\psi_x a; a \in P^1(A)\}$ is also weakly compact, see [7]. Consequently ψ_x is weakly almost periodic.

Now, let m be a topological left invariant mean on $WAP(A)$. Define ϕ on X by $\langle \phi, x \rangle = \langle m, \psi_x \rangle$ for every $x \in X$. An argument similar to that in the proof of Theorem 3.4, shows that $\|\varphi\| = \|\phi\|$ and $\langle \phi, T_a(x) \rangle = \langle \phi, x \rangle$ for every $x \in X$ and $a \in P^1(A)$. This completes our proof. □

Proposition 3.6. *Let A be a left amenable Lau algebra. Let T be a separately weakly continuous action of A on a Banach space X such that*

$\|T_a\| \leq 1$ for each $a \in P^1(A)$. If $x \in X$ and $d(o, O_x) > 0$, then the distance from the origin to O_x in X is given by $d(0, O_x) = |\langle \phi, x\rangle|$ where ϕ is an element of norm 1 in X^* and $\langle \phi, T_a(x)\rangle = \langle \phi, x\rangle$ for each $x \in X$ and $a \in P^1(A)$.

Proof. Suppose that $x \in X$. Let $c = dist(0, O_x) > 0$. As a consequence of the Hahn-Banach theorem there is an element $\psi \in X^*$ of norm 1 for which $c \leq |\langle \psi, y\rangle|$ for all $y \in O_x$. The mapping $\psi a : y \mapsto \psi(T_a y)$ belongs to X^*. Let O_x^* be the weak* closure of $\left\{\psi a; a \in P^1(A)\right\}$ with its weak*-topology. By Theorem 4.14 in [23], O_x^* is weak*-compact. It is not difficult to see that $c \leq |\langle \varphi, y\rangle|$ for each $y \in O_x$ and $\varphi \in O_x^*$. Thus $c \leq \|\varphi\| \|y\|$, and so $c \leq \|\varphi\| c$ for all $\varphi \in O_x^*$. This shows that $1 \leq \|\varphi\|$, and so $\|\varphi\| = 1$ for all $\varphi \in O_x^*$. If m is any topologically left invariant mean on A^*, the weak* density of $P^1(A)$ in the set of all means on A ensures that we can find weak* convergent net $\{a_\alpha\}$ such that $a_\alpha \to m$ [7]. Then for every $f \in A^*$ and every $a \in P^1(A)$, $\langle f, aa_\alpha - a_\alpha\rangle \to 0$. This means that $\{a_\alpha a - a_\alpha\}$ converges to 0 in the weak topology. Consider the net $\{\psi a_\alpha\}$. By compactness of O_x^*, we can assume $\psi a_\alpha \to \phi$ in O_x^*, passing to a subnet if necessary. If $a \in A$ and $y \in X$, we have

$$\begin{aligned}\langle \phi, T_a(y)\rangle &= \lim_\alpha \langle \psi a_\alpha, T_a(y)\rangle \\ &= \lim_\alpha \langle \psi, T_{aa_\alpha}(y)\rangle \\ &= \lim_\alpha \langle \psi, T_{a_\alpha}(y)\rangle \\ &= \lim_\alpha \langle \psi a_\alpha, y\rangle \\ &= \langle \phi, y\rangle .\end{aligned}$$

But ϕ is constant on O_x, since $\langle \phi, T_a(x)\rangle = \langle \phi, x\rangle$ for $a \in P^1(A)$. Moreover,

$$\begin{aligned}c &\leq |\langle \phi, x\rangle| \\ &= |\langle \phi, y\rangle| \\ &\leq \|y\| \|\phi\| \\ &= \|y\|\end{aligned}$$

for all y in O_x. Since $\|y\|$ can be chosen arbitrarily, then $|\langle \phi, x\rangle| = c$. □

Let A be a Lau algebra, $Q^1(A)$ a closed subalgebra of $P^1(A)$. We denote by $N(A^*, Q^1(A))$ the vector space spanned by

$$\left\{f - fa; f \in A^*, a \in Q^1(A)\right\}.$$

We consider the set of $Q^1(A)$-invariant means on A^*,

$$\mathcal{M}\left(A^*, Q^1(A)\right) = \left\{m \in \mathcal{M}; \langle m, fa\rangle = \langle m, f\rangle \text{ for all } f \in A^*, a \in Q^1(A)\right\}.$$

If $\mathcal{M}\left(A^*, Q^1(A)\right) \neq \emptyset$, we define the mapping

$$S(f) = \sup\left\{\operatorname{Re}\langle m, f\rangle, m \in \mathcal{M}\left(A^*, Q^1(A)\right)\right\}$$

for every $f \in A^*$, we have $S(f) \leq \|f\|$.

Proposition 3.7. *Let A be a Lau algebra, $Q^1(A)$ a closed subalgebra of $P^1(A)$. If $\mathcal{M}\left(A^*, Q^1(A)\right) \neq \emptyset$, then for every $f \in A^*$,*

$$S(f) = \inf\left\{\sup\left\{\operatorname{Re}\langle f+h, a\rangle; a \in P^1(A)\right\}, h \in N\left(A^*, Q^1(A)\right)\right\}.$$

Proof. If $f \in A^*$, let

$$\rho(f) = \inf\left\{\sup\left\{\operatorname{Re}\langle f+h, a\rangle; a \in P^1(A)\right\}, h \in N\left(A^*, Q^1(A)\right)\right\}.$$

Obviously ρ is positively homogeneous.. If $f_1, f_2 \in A^*$, for every $\epsilon > 0$, there exist $h_1, h_2 \in N\left(A^*, Q^1(A)\right)$ such that

$$\rho(f_i) > \sup\left\{\operatorname{Re}\langle f_i + h_i, a\rangle; a \in P^1(A)\right\} - \epsilon \quad \text{for} \quad i = 1, 2.$$

So $\rho(f_1) + \rho(f_2) > \rho(f_1 + f_2) - 2\epsilon$. As $\epsilon > 0$ may be chosen arbitrarily, we must have $\rho(f_1) + \rho(f_2) \geq \rho(f_1 + f_2)$. This shows that ρ is subadditive.

Fix $f_0 \in A^*$. Put $M_0 = \{\alpha f_0; \alpha \in \mathbb{R}\}$. Define F on M_0 by $\langle F, \alpha f_0\rangle = \alpha\rho(f_0)$. Then $\langle F, \alpha f_0\rangle \leq \alpha\rho(f_0) = \rho(\alpha f_0)$ for $\alpha > 0$, and F is real linear on M_0. Consider A^* as a real vector space and apply the Hahn–Banach theorem for real vector spaces to the real-linear functional F to obtain a real-linear extension $m_0 : A^* \to \mathbb{R}$ that is also dominated above by ρ, so that it satisfies $-\rho(-f) \leq \langle m_0, f\rangle \leq \rho(f)$ for all $f \in A^*$ [23]. If $\{f_n\}$ is a sequence in A^* such that $f_n \to 0$, then $\rho(f_n) \to 0$ and $\rho(-f_n) \to 0$. Since $-\rho(-f_n) \leq \langle m_0, f_n\rangle \leq p(f_n)$, we have $\langle m_0, f_n\rangle \to 0$. Hence $m_0 \in A^{**}$. For every $f \in A^*$,

$$\begin{aligned}\langle m_0, f\rangle &\leq \sup\left\{\operatorname{Re}\langle f+0, a\rangle; a \in P^1(A)\right\} \\ &\leq \|f\|.\end{aligned}$$

On the other hand, $1 \leq -\rho(-1) \leq \langle m_0, 1\rangle \leq \rho(1) \leq 1$, that is, m_0 is a mean on A^*. For every $h \in N\left(A^*, Q^1(A)\right)$, $\langle m_0, h\rangle \leq \rho(h) \leq 0$, and $-\langle m_0, h\rangle = \langle m_0, -h\rangle \leq \rho(-h) \leq 0$, and so $\langle m_0, fa\rangle = \langle m_0, f\rangle$ for all $f \in A^*$ and $a \in Q^1(A)$.

Assume that the scalar field is $\mathbb{C}$. Define $m : A^* \to \mathbb{C}$ by

$$\langle m, f\rangle = \langle m_0, f\rangle - i\langle m_0, if\rangle.$$

Then m is complex linear. It is known that a complex linear functional on A^* is in A^{**} if its real part is continuous. Thus $m \in A^{**}$. Since $\langle m_0, i1\rangle \leq \sup\left\{\operatorname{Re}\langle i1, a\rangle; a \in P^1(A)\right\} = 0$, and

$$\begin{aligned}-\langle m_0, i1\rangle &= \langle m_0, -i1\rangle \\ &\leq \sup\left\{\operatorname{Re}\langle -i1, a\rangle; a \in P^1(A)\right\} \\ &= 0,\end{aligned}$$

we have $\langle m_0, i1\rangle = 0$, where i denotes the imaginary unit. This shows that $\langle m, 1\rangle = 1$.

Let $f \in A^*$. There exists $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and

$$\begin{aligned}\langle m, \alpha f\rangle &= \alpha \langle m, f\rangle \\ &= |\langle m, f\rangle| .\end{aligned}$$

Hence $|\langle m, f\rangle| = \langle m, \alpha f\rangle = \langle m_0, \alpha f\rangle \leq \|f\|$, and therefore $\|m\| = \langle m, 1\rangle = 1$. Since $\langle m_0, fa\rangle = \langle m_0, f\rangle$ for all $f \in A^*$ and $a \in Q^1(A)$, we have $m \in \mathcal{M}\left(A^*, Q^1(A)\right)$. Therefore

$$\begin{aligned}\rho(f_0) &= \langle m_0, f_0\rangle \\ &= \mathrm{Re}\,\langle m, f_0\rangle \\ &\leq S(f_0) .\end{aligned}$$

Now, let $m \in \mathcal{M}\left(A^*, Q^1(A)\right)$. Let $\{a_\alpha\}$ be a net in $P^1(A)$ such that $\{a_\alpha\}$ converges to m in the weak* topology of A^*. For every $h \in N\left(A^*, Q^1(A)\right)$,

$$\begin{aligned}\mathrm{Re}\,\langle m, f_0\rangle &= \mathrm{Re}\,\langle m, f_0 + h\rangle \\ &= \mathrm{Re}\lim_\alpha \langle f_0 + h, a_\alpha\rangle \\ &\leq \sup\left\{\mathrm{Re}\,\langle f_0 + h, a\rangle ; a \in P^1(A)\right\},\end{aligned}$$

hence $S(f_0) \leq \sup\left\{\mathrm{Re}\,\langle f_0 + h, a\rangle ; a \in P^1(A)\right\}$. Thus $S(f_0) \leq \rho(f_0)$. Consequently $\rho = S$. □

Assume that G is amenable as discrete. Let $\mathcal{F}$ be the family of all finite subsets of G. For every nonvoid finite subset F of G,

$$f + (cardF)^{-1} \sum_{a\in\mathcal{F}} ((-f) - {}_a(-f)) = (cardF)^{-1} \sum_{a\in\mathcal{F}} {}_a f;$$

hence by Proposition 3.7, we have

$$S(f) \leq \inf\left\{ess\sup\left[(cardF)^{-1} \sum_{a\in\mathcal{F}} {}_a f\right]; F \in \mathcal{F}\right\}$$

for all $f \in l^\infty(G)^+$.

Let A be a Lau algebra. We define $V\left(A^*, P^1(A)\right)$ to be the set of all $f \in A^*$ such that, for every $m \in \mathcal{M}\left(A^*, P^1(A)\right)$, $m(f)$ admits a constant value $v(f)$. Let G be a discrete group. Miao proved in [20] that G is amenable if and only if $V\left(l^1(G)^*, P^1\left(l^1(G)\right)\right)$ is closed under addition.

Proposition 3.8. *Let A be a left amenable Lau algebra. Then*

$$V\left(A^*, P^1(A)\right) = \mathbb{R}1 \oplus \overline{N\left(A^*, P^1(A)\right)}.$$

Proof. Let $m \in TLIM\,(A^*)$. It is not difficult to see that

$$\mathbb{R}1 \oplus \overline{N\,(A^*, P^1\,(A))} \subseteq V\left(A^*, P^1\,(A)\right).$$

Moreover $\overline{N\,(A^*, P^1\,(A))} \bigcap \mathbb{R}1 = \{0\}$. Indeed, let

$$c1 \in \mathbb{R}1 \bigcap \overline{N\,(A^*, P^1\,(A))}$$

and $\{f_n\}$ a sequence in $N\left(A^*, P^1\,(A)\right)$ such that $f_n \to c1$. Since m is continuous, $0 = \langle m, f_n\rangle \to \langle m, c1\rangle = c$.

Reverse inclusion: If $f_0 \in V\left(A^*, P^1\,(A)\right)$ and $v(f_0) = c$, then

$$v\,(f_0 - c1) = 0.$$

If $f_0 - c1 \notin \overline{N\,(A^*, P^1\,(A))}$, by the Hahn Banach theorem [23] there exists $F \in A^{**}$ such that $\langle F, f_0 - c1\rangle \neq 0$ and $\langle F, f\rangle = 0$ whenever $f \in N\left(A^*, P^1\,(A)\right)$. It follows that $aF = F$ for all $a \in P^1\,(A)$. Hence $aF = F$ and $aF^* = F^*$ for all $a \in P^1\,(A)$. Without loss of generality, we may assume that F is self adjoint. Write $F = F^+ - F^-$, the orthogonal decomposition of F [7]. We can assume that $F^+ \neq 0$ and $F^- \neq 0$. Proof of the case $F^+ = 0$ or $F^- = 0$ is similar. Then

$$m_1 := \frac{F^+}{F^+\,(1)}, m_2 := \frac{F^-}{F_1^-\,(1)} \in \mathcal{M}.$$

Since $F = 0$ on $N\left(A^*, P^1\,(A)\right)$, $mF = F$. Indeed, let $\{a_\alpha\}$ be a net in $P^1\,(A)$ such that $\{a_\alpha\}$ converges to m in the weak* topology [16]. For every $f \in A^*$,

$$\begin{aligned}\langle mF - F, f\rangle &= \lim_\alpha \langle a_\alpha F - F, f\rangle \\ &= \lim_\alpha \langle F, f a_\alpha - f\rangle \\ &= 0.\end{aligned}$$

If $a \in P^1\,(A)$, then $aF^+ - aF^- = aF = F = F^+ - F^-$. Since aF^+, aF^- are positive, and

$$\begin{aligned}\left\|aF^+\right\| + \left\|aF^-\right\| &= \left\langle aF^+, 1\right\rangle + \left\langle aF^-, 1\right\rangle \\ &= \left\langle F^+, 1\right\rangle + \left\langle F^-, 1\right\rangle \\ &= \|F\|,\end{aligned}$$

it follows that $aF^+ = F^+$ and $aF^- = F^-$ [7]. Consequently $am_1 = m_1$ and $am_2 = m_2$ for all $a \in P^1\,(A)$. It is easy to see that $mm_1 = m_1$ and also $mm_2 = m_2$. On the other hand,

$$\begin{aligned}\langle mF, f_0 - c1\rangle &= F^+\,(1)\,\langle mm_1, f_0 - c1\rangle - F^-\,(1)\,\langle mm_2, f_0 - c1\rangle \\ &= F^+\,(1)\,\langle m_1, f_0 - c1\rangle - F^-\,(1)\,\langle m_2, f_0 - c1\rangle \\ &= F^+\,(1)\,(v\,(f_0) - c) - F^-\,(1)\,(v\,(f_0) - c)\end{aligned}$$

$$= 0.$$

Therefore $\langle F, f_0 - c1\rangle = \langle mF, f_0 - c1\rangle = 0$. We come to a contradiction. So we must have $f_0 - c1 \in \overline{N(A^*, P^1(A))}$. Consequently

$$V\left(A^*, P^1(A)\right) = \mathbb{R}1 \oplus \overline{N(A^*, P^1(A))}. \qquad \square$$

Corollary 3.9. *Let A be a Lau algebra. Then A is left amenable if and only if*

$$\inf\left\{\sup\left\{\mathrm{Re}\langle 1 - f, a\rangle ; a \in P^1(A)\right\}, f \in N\left(A^*, P^1(A)\right)\right\} = 1.$$

Proof. Let A be a left amenable Lau algebra, and let $m \in TLIM(A)$. For every $f \in N\left(A^*, P^1(A)\right)$,

$$\begin{aligned}\sup\left\{\mathrm{Re}\langle 1 - f, a\rangle ; a \in P^1(A)\right\} &\geq \mathrm{Re}\langle m, 1 - f\rangle \\ &= 1.\end{aligned}$$

For the null functional f_0 on A, $f_0 \in N\left(A^*, P^1(A)\right)$ and so

$$\inf\left\{\sup\left\{\mathrm{Re}\langle 1 - f, a\rangle ; a \in P^1(A)\right\}, f \in N\left(A^*, P^1(A)\right)\right\} = 1.$$

Now, let $\inf\left\{\sup\left\{\mathrm{Re}\langle 1 - f, a\rangle ; a \in P^1(A)\right\}, f \in N\left(A^*, P^1(A)\right)\right\} = 1$. We show that $TLIM(A) \neq \emptyset$. Let $f_0 \in A^* \setminus N\left(A^*, P^1(A)\right)$. By the Hahn-Banach theorem there exists a continuous linear functional F on A^* such that $\langle F, f_0\rangle = 1$ and $F = 0$ on $N\left(A^*, P^1(A)\right)$ [23]. Without loss of generality, we may assume that F is self adjoint. There exist nonnegative linear functionals F^+ and F^- on A^* such that $F = F^+ - F^-$ and $aF^+ = F$, $aF^- = F^-$ for all $a \in P^1(A)$. As $\langle F, f_0\rangle = 1$, we must have $|\langle F^+, f_0\rangle| + |\langle F^-, f_0\rangle| > 0$, say $|\langle F^+, f_0\rangle| > 0$ for instance. As $|\langle F^+, f_0\rangle| \leq \langle F^+, 1\rangle \|f_0\|$, we have $|\langle F^+, 1\rangle| > 0$. Define $\langle F^+, 1\rangle \langle m, f\rangle = \langle F^+, f\rangle$. So $m \in TLIM(A)$. Consequently A is left amenable if and only if

$$\inf\left\{\sup\left\{\mathrm{Re}\langle 1 - f, a\rangle ; a \in P^1(A)\right\}, f \in N\left(A^*, P^1(A)\right)\right\} = 1. \qquad \square$$

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