

On l^1 -Munn Projective Tensor Product Algebras and Connes Amenability

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ABSTRACT. In this paper, we study and investigate Connes amenability for l^1 -Munn algebra $\mathcal{LM}(\widehat{\mathcal{A}}\widehat{\otimes}\widehat{\mathcal{B}}; P, I, J)$ where \mathcal{A} and \mathcal{B} are Banach algebras, I and J are nonempty sets and P is invertible matrix. Then we use the obtained results to semigroup algebras $l^1(S\widehat{\otimes}T)$ that S and T are semigroups. Also, we prove that if $S\widehat{\otimes}T$ is a weakly cancellative semigroup and $l^1(S\widehat{\otimes}T)$ be Connes amenable then the idempotents set of $S\widehat{\otimes}T$ will be finite.

Keywords: Connes amenability, Banach algebras, l^1 -Munn algebras, semigroup algebras, projective tensor product

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1. Introduction

l^1 -Munn algebras are introduced by Eslamzadeh in [?]. Eslamzadeh characterized amenable semigroup algebras by l^1 -Munn algebras. In [?], Munn introduced a certain type of mentioned algebras. l^1 -Munn algebras has been investigated in some texts. For instance, Blackmore showed the l^1 -Munn algebra of the group algebra $l^1(G)$ is weakly amenable. Also, the structure of this algebras studied by Eslamzadeh in [?]. On the other hand, the authors in [?] used the l^1 -Munn algebras to study of semigroup algebras of completely simple semigroups. We know that special concept of amenability was called Connes amenability In [?], Runde extended the notion of Connes-amenability to dual Banach algebras. For a locally compact group G , the group algebra $l^1(G)$ and the measure algebra $M(G)$ are two examples of dual Banach algebras. Runde introduced normal, virtual diagonals for a dual Banach algebra and showed that the existence of a normal virtual diagonal for $M(G)$ is equivalent to it being Connes amenable. In particular, $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable. φ -Connes module amenability of dual Banach algebras, Connes amenability for certain product of Banach algebras and Connes amenability of l^1 -Munn Algebras are investigated in [?, ?, ?]. In this paper we investigate the semigroup algebra of projective tensor product of weakly cancellative semigroups in theorem of Runde. Also, the concept of Connes amenability of l^1 -Munn Projective Tensor Product algebras is investigated. We apply the l^1 -Munn algebras to study of Connes amenability of mentioned Projective Tensor Product of semigroup algebras.

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2. Main results

Let \mathcal{A} be a dual Banach algebra such that $(\mathcal{A}_*)^* = \mathcal{A}$. A dual Banach \mathcal{A} -bimodule X is called normal Banach \mathcal{A} -bimodule if for each $x \in X$, the maps $a \rightarrow a.x$, $a \rightarrow x.a$ are ω^* -continuous ($a \in \mathcal{A}$).

Now, in the following we present some definitions.

DEFINITION 2.1. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and X be a Banach \mathcal{A} -bimodule. The linear bounded map $D : \mathcal{A} \rightarrow X$ is called derivation if for every $a, b \in \mathcal{A}$ we have $D(ab) = D(a).b + a.D(b)$. We say D is inner derivation if there exists $x \in X$ such that for every $a \in \mathcal{A}$, $D(a) = a.x - x.a$. In this case we denote D by ad_x .

Let \mathcal{A} be a Banach algebra and X be a Banach \mathcal{A} -bimodule. \mathcal{A} is called amenable if every derivation $D : \mathcal{A} \rightarrow X^*$ is inner.

DEFINITION 2.2. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and X be a normal dual Banach \mathcal{A} -bimodule. \mathcal{A} is called Connes amenable if every ω^* -continuous derivation $D : \mathcal{A} \rightarrow X$ is inner.

REMARK 2.3. Let X be a Banach \mathcal{A} -bimodule. An element $x \in X$ is called ω^* -weakly continuous if the module maps $a \rightarrow x.a, a \rightarrow a.x$ are ω^* -weak continuous ($a \in \mathcal{A}$). The collection of all ω^* -weakly continuous elements of X is denoted by $\sigma wc(X)$. We define the map $\pi : \widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{A}}$ by $\pi(a \otimes b) = ab$, ($a, b \in \widehat{\mathcal{A}}$). Then we have $\pi^* : \mathcal{A}_* \rightarrow \sigma wc((\widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}})^*)$. Consequently, π^{**} drops to a homomorphism $\pi_{\sigma wc} : \sigma wc((\widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}})^*)^* \rightarrow \mathcal{A}$.

DEFINITION 2.4. An element $M \in \sigma wc((\widehat{\mathcal{A}} \otimes \widehat{\mathcal{A}})^*)^*$ is called a σwc -virtual diagonal for \mathcal{A} , if $M.u = u.M$ for every $u \in \mathcal{A}$ and $u.\pi_{\sigma wc}(M) = u$ for every $u \in \mathcal{A}$.

It is shown that Connes amenability of Banach algebra \mathcal{A} is equal to existence a σwc -virtual diagonal for \mathcal{A} .

DEFINITION 2.5. Let \mathcal{A} be a unital Banach algebra, let $I \neq \emptyset, J \neq \emptyset$ be arbitrary sets and $P = (p_{ij}) \in M_{J \times I}(\mathcal{A})$ be a matrix such that $\|P\|_\infty = \sup\{\|p_{ji}\| : j \in J, i \in I\} \leq 1$. The set $M_{I \times J}(\mathcal{A})$ of all $I \times J$ matrices $a = (a_{ij})$ on \mathcal{A} with l^1 -norm and the product $A \bullet B = APB$, ($A, B \in M_{I \times J}(\mathcal{A})$) is a Banach algebra that is called l^1 -Munn algebra on \mathcal{A} with sandwich matrix P . In this case l^1 -Munn algebra is denoted by $\mathcal{LM}(\mathcal{A}; P, I, J)$ [?].

The following statements are from [?].

A semigroup S is called regular if for every $a \in S$ there is $b \in S$ such that $a = aba$. S is an inverse semigroup if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$. Let G be a group, $I \neq \emptyset$ and $J \neq \emptyset$ be arbitrary sets, and $G^0 = G \cup \{0\}$ be the group with zero arising from G by adjunction of a zero element. An $I \times J$ matrix A over G^0 that has at most one nonzero entry $a = A(i, j)$ is called a Rees $I \times J$ matrix over G^0 and is denoted by $(a)_{ij}$. Let P be a $J \times I$ matrix over G^0 . The set $S = G \times I \times J$ with the composition $(a, i, j) \circ (b, l, k) = (aP_{jl}b, i, k)$, $(a, i, j), (b, l, k) \in S$ is a semigroup that we denote by $\mathcal{M}(G, P)$ [?, p. 68]. Similarly if P is a $J \times I$ matrix over G^0 , then $S = G \times I \times J \cup \{0\}$ is a semigroup under the following composition operation:

$$(a, i, j) \circ (b, l, k) = \begin{cases} (aP_{jl}b, i, k); & P_{jl} \neq 0 \\ 0; & P_{jl} = 0 \end{cases}$$

$$(a, i, j) \circ 0 = 0 \circ (a, i, j) = 0 \circ 0 = 0$$

Mentioned semigroup which is denoted by $\mathcal{M}^0(G, P)$ also can be described in the following way: The set of all Rees $I \times J$ matrices over G^0 form a semigroup under the binary operation $A \bullet B = APB$, which is called the Rees $I \times J$ matrix semigroup over G^0 with the sandwich matrix P and is isomorphic to $\mathcal{M}^0(G, P)$ [?, pp. 61-63]. An $I \times J$ matrix P over G^0 is called regular (invertible) if every row and every column of P contains at least (exactly) one nonzero entry.

Already in [?], Esslamzadeh shows that if $\mathcal{LM}(\mathcal{A}; P, I, J)$ is amenable then \mathcal{A} is amenable, $I \neq \emptyset$ and $J \neq \emptyset$ are finite and P is invertible. In the following theorem we investigate this subject for Connes amenability of a projective tensor product Banach algebra.

THEOREM 2.6. *Let \mathcal{A} and \mathcal{B} be dual Banach algebras. Let $\mathcal{LM}(\mathcal{A}; P, I, J)$ and $\mathcal{LM}(\mathcal{B}; P, I, J)$ be Connes amenable, then $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is Connes amenable. Moreover the sets I and J are finite and matrix P is invertible.*

PROOF. Let $\mathcal{LM}(\mathcal{A}; P, I, J)$ and $\mathcal{LM}(\mathcal{B}; P, I, J)$ be Connes amenable. Thus they have bounded approximate identities, say (e_α) and (e_β) , respectively. Thus $\mathcal{A} \widehat{\otimes} \mathcal{B}$ has bounded approximate identity [?]. By applying [?, Lemma 3.7], we imply that I and J are finite sets. So, P must be invertible. For prove Connes amenability of $\mathcal{A} \widehat{\otimes} \mathcal{B}$, it is sufficient that suppose $X = (X_*)^*$ be a normal Banach $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -bimodule and $D : \mathcal{A} \widehat{\otimes} \mathcal{B} \rightarrow X$ be a ω^* -continuous derivation. One can see that D is inner. \square

If in the above theorem, \mathcal{A} has an identity element then we conclude the following corollary.

COROLLARY 2.7. *Let \mathcal{A} and \mathcal{B} be Connes amenable unital dual Banach algebras, I and J are finite sets and P is invertible matrix. Then Banach algebra $\mathcal{LM}(\mathcal{A} \widehat{\otimes} \mathcal{B}; P, I, J)$ is Connes amenable.*

PROOF. Without losing the generality let $|I| = |J| = n$. Suppose that X be a normal Banach $\mathcal{LM}(\mathcal{A} \widehat{\otimes} \mathcal{B}; I_n, I, J) \widehat{\otimes} \mathcal{A} \widehat{\otimes} \mathcal{B}$ -bimodule and $D : \mathcal{LM}(\mathcal{A} \widehat{\otimes} \mathcal{B}; I_n, I, J) \widehat{\otimes} \mathcal{A} \widehat{\otimes} \mathcal{B} \rightarrow X$ be a ω^* -continuous derivation. Now, we define the right and left module actions on $\mathcal{A} \widehat{\otimes} \mathcal{B}$, then we prove that D is inner. This complete the proof. \square

Let S be a semigroup. We say that S is cancellative semigroup, if for every $r, p \neq q \in S$ we have $rs \neq rq$ and $pr \neq qr$.

An element $e \in S$ is called an idempotent if $e = e^* = e^2$. For $p \in S$, we define $L_p, R_p : S \rightarrow S$ by $L_p(q) = pq, R_p(q) = qp; (q \in S)$. If for each $p \in S$, L_p and R_p are finite-to-one maps, in this case S is named weakly cancellative. Suppose that S be a semigroup. S is called simple semigroup if only ideal in S is itself. S is called completely simple semigroup if it is simple and contain a principle idempotent.

LEMMA 2.8. *Let S and T be weakly cancellative semigroups. Let $l^1(S \widehat{\otimes} T)$ be Connes amenable unital semigroup algebra. Then the idempotents set $S \widehat{\otimes} T$ is finite.*

THEOREM 2.9. *Let G be a group, S and T be weakly cancellative semigroups. Let $l^1(S \widehat{\otimes} T)$ be Connes amenable unital semigroup algebra. Then $S \widehat{\otimes} T$ is a Rees matrix semigroup of the form $S \widehat{\otimes} T = \mathcal{M}(G; P, I, J)$ and l^1 -Munn algebra $\mathcal{LM}(l^1(G); P, I, J)$ has an identity.*

PROOF. By Lemma ??, $S \widehat{\otimes} T$ is a simple semigroup with finite idempotents set, thus $S \widehat{\otimes} T$ is a completely simple semigroup. Therefore, $S \widehat{\otimes} T$ is a Rees matrix semigroup of the form $S \widehat{\otimes} T = \mathcal{M}(G; P, I, J)$. This complete the proof. \square

THEOREM 2.10. *Let $S\widehat{\otimes}T$ be a weakly cancellative semigroup with finite idempotents set. Let $l^1(S\widehat{\otimes}T)$ be an unital semigroup algebra. Moreover, suppose that $S\widehat{\otimes}T$ be a Rees matrix semigroup of the form $S\widehat{\otimes}T = \mathcal{M}(G; P, I, J)$. If $l^1(S\widehat{\otimes}T)$ is Connes amenable then $l^1(S\widehat{\otimes}T)$ is amenable and vice versa.*

PROOF. If $l^1(S\widehat{\otimes}T)$ is Connes amenable then by Theorem ?? and [?] the proof is clearly. The converse follows directly from Theorem ??. \square

3. Conclusion

In this paper, we obtained the relation between Connes amenability and amenability of l^1 -Munn projective tensor product algebras. Also, we present some characterization for Rees matrix semigroup algebras.

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References

1. J. Duncan and A. L. T. Paterson, *Amenability for discrete convolution semigroup algebras*, Math. Scand., 66 (1990), 141–146.
2. G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications*, J. Funct. Anal., 161 (1999), 364–383.
3. G. H. Esslamzadeh, *Representation theory and positive functionals of involutive l^1 -Munn algebras*, Semigroup Forum, 69 (2004), 51–62.
4. A. Ghaffari, S. Javadi and E. Tamimi, *Connes amenability for certain product of Banach algebras*, Wavelet and Linear Algebra. 9 (1) (2022), 1-14.
5. A. Ghaffari, S. Javadi and E. Tamimi, *Connes Amenability of l^1 -Munn Algebras*, Tamkang Journal of Mathematics. 53 (2022), 259-266.
6. A. Ghaffari, S. Javadi and E. Tamimi, *φ -Connes module amenability of dual Banach algebras*, Journal of Algebraic Systems. 8 (1) (2020), 69-82.
7. O. T. Mewomo and O. J. Ogunsola, *On n -weak amenability of semigroup algebras*, J. Nigerian Math. Soc., 32 (2013), 298–301.
8. W. D. Munn, *On semigroup algebras*, Math. Proc. Cambridge Philos. Soc., 51 (1955), 1–15.
9. V. Runde, *Amenability for dual Banach algebras*, Studia Math., 148 (2001), 47–66.
10. V. Runde, *Lectures on Amenability, Lecture Notes in Mathematics* (Springer-Verlag, Berlin, 2002).