ISC
Islamic World Science Citation Center
04250- 38506
2nd
International Conference on Mathematics and Its Applications
August 20-21,2025, Shahid Chamran University of Ahvaz, Ahvaz, IRAN
Shahid chanran University of Ahvaz

On φ-Splitting of the Short Exact Sequences of Dual Banach Algebras

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Abstract

Let φ and ψ be w^*-continuous non-zero linear functionals on dual Banach algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. In this paper, the relation between the notions of φ-splitting of the short exact sequences, are given. In addition to this, some examples of dual Banach algebras which confirm the mentioned properties are presented.

Keywords: φ-σwc virtual diagonal, dual Banach algebra, φ-splitting, the short exact sequences.

1 Introduction

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach $\mathcal{A}$-bimodule. Then E is called dual if there exists a closed submodule $E_* \subseteq E^*$, such that $E = (E_*)^*$. A dual Banach $\mathcal{A}$-bimodule E is called normal if the module actions of $\mathcal{A}$ on E are w^*-continuous. If a Banach algebra as a Banach $\mathcal{A}$-bimodule is dual, then it is called dual. If $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$, then we write $\mathcal{A} = (\mathcal{A}_*)^*$, see [5, 6].

Let $\mathcal{A}$ and E be as above. A bounded, linear map $D : \mathcal{A} \to E$ satisfying $D(ab) = a.D(b) + D(a).b$ for every $a, b \in \mathcal{A}$ is called derivation. A derivation D is inner if there exists $x \in E$ such that for every $a \in \mathcal{A}$, we have $D_x(a) = a.x - x.a$, see [14].

Splitting and admissibility of the short exact sequences are significant notions in homology theory of Banach algebras. Let $\mathcal{A}$ be a Banach algebra, and let $3 \leq m \in \mathbb{N}$. A sequence

$$\mathcal{S}_1 \overset{\varphi_1}{\to} \mathcal{S}_2 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{m-1}}{\to} \mathcal{S}_m \tag{1.1}$$

of $\mathcal{A}$-bimodules $S_1, S_2, \cdots, S_m$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_j : \mathcal{S}_j \to \mathcal{S}_{j+1}$ for $j \in \{1, \cdots, m-1\}$ is called exact at position $j = 2, \cdots, m-1$ if $Im\, \varphi_{j-1} = \ker \varphi_j$. If the sequence in (1.1), is exact at every position $j \in \{2, \cdots, m-1\}$, then it is called exact. Let $\mathcal{A}$ be a Banach algebra. A short exact sequence

$$\Theta : 0 \to \mathcal{S}_1 \overset{\varphi_1}{\to} \mathcal{S}_2 \overset{\varphi_2}{\to} \cdots \overset{\varphi_{m-1}}{\to} \mathcal{S}_m \to 0 \tag{1.2}$$

of Banach $\mathcal{A}$-bimodules $\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_m$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_j : \mathcal{S}_j \to \mathcal{S}_{j+1}$ for $j \in \{1, \cdots, m-1\}$, is admissible, if there exists a bounded linear map $\rho_j : \mathcal{S}_{j+1} \to \mathcal{S}_j$ such that $\rho_j \circ \varphi_j$ on $\mathcal{S}_j$ for $j = 1, 2, \cdots, m-1$, is the identity map. Further we say Θ, splits if we may choose ρ_j to be an $\mathcal{A}$-bimodule homomorphism. Note that if $\mathcal{A}$ is a Banach algebra then the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach

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$\mathcal{A}$-bimodule in the canonical way. In this paper, we consider the following $\mathcal{A}$-bimodule homomorphism (projection induced product mapping)

$$\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}; \qquad a \otimes b \mapsto ab, \qquad (a, b \in \mathcal{A}). \tag{1.3}$$

Assume now that $\mathcal{A}$ is a unital Banach algebra, then the short exact sequence of Banach $\mathcal{A}$-bimodules, $0 \to \ker \pi \to \mathcal{A}\widehat{\otimes}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0$, is admissible. It is known that the splitting of the short exact sequences is playing important role in harmonic analysis; in particular, in the notion of Connes amenability of Banach algebras. For instance, in [2, Proposition 4.4], Daws showed that the notion of Connes amenability of a Banach algebra is equivalent to the splitting of the related short exact sequence. Also, Chen investigated the splitting of Rees short exact sequences in [1]. Indeed, a Banach algebra $\mathcal{A}$ is called Connes amenable, if for every normal Banach $\mathcal{A}$-bimodule E, every w^*-continuous derivation $D : \mathcal{A} \to E$ is inner, see [14, Definition 4.4.7]. Connes amenability of l^1- Munn algebras is studied by Ghaffari et al. [7]. Runde showed that if G is a locally compact group, then Connes amenability of measure algebra $M(G)$, the existence of a normal virtual diagonal for $M(G)$ and amenability of G are all equivalent, (see [14, Definition 4.4.14] and [13]).
We remark some standard notations and definitions that we shall need in this paper. Throughout the paper, $\Delta(\mathcal{A})$ and $\Delta_{w^*}(\mathcal{A})$ will denote the character space of all continuous and w^*-continuous non-zero complex linear functionals on Banach algebra $\mathcal{A}$, respectively. If $\mathcal{A} = (\mathcal{A}_*)^*$ is a dual Banach algebra and E is a Banach $\mathcal{A}$-bimodule, then $\sigma wc(E)$, a closed submodule of E, denotes the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \to E; \qquad a \mapsto a.x, \quad a \mapsto x.a, \tag{1.4}$$

are w^*-w continuous. The Banach $\mathcal{A}$-bimodules E that are related to such E are called Banach φ-bimodules, where $\varphi \in \Delta_{w^*}(\mathcal{A})$.

Remark 1.1. Suppose that $\mathcal{A}$ is a dual Banach algebra. Then by above notations, $\Delta_{w^*}(\mathcal{A}) = \Delta(\mathcal{A}) \cap \sigma wc(\mathcal{A}^*)$, see [11, Lemma 2.3]. Indeed, throughout of the paper we use $\Delta_{w^*}(\mathcal{A})$ instead $\Delta(\mathcal{A}) \cap \sigma wc(\mathcal{A}^*)$.

Runde characterized Connes amenability of dual Banach algebras via the existence of normal virtual diagonals, see [14, Theorem 4.4.15]. Also, module Connes amenability for projective tensor product and Θ-Lau product of Banach algebras, χ-module normal virtual diagonal for semigroup algebras and $\chi \otimes \eta$-strong Connes amenability of certain dual Banach algebras, where χ and η are bounded w^*-continuous module homomorphisms from a semigroup algebra to itself is studied in [17, 18, 19].

Dual Banach algebra $\mathcal{A}$ is called φ-Connes amenable if for every normal φ-bimodule E, every bounded w^*-continuous derivation $D : \mathcal{A} \to E$ is inner, where $\varphi \in \Delta_{w^*}(\mathcal{A})$, see [4, Definition 2.1]. The notion of φ-Connes amenability of dual Banach algebras is investigated by Ghaffari and Javadi in [4]. Also, Connes amenability for certain product of Banach algebras and φ-Connes module amenability of dual Banach algebras, where φ is a w^*-continuous bounded module homomorphism from a Banach algebra on itself is investigated by Ghaffari et al., see [5, 6].

Helemskii investigated the homological properties of Banach algebras. He presented the concepts of biflat Banach algebras. Indeed, let $\mathcal{A}$ be a Banach algebra and $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ be the projection induced product map in (2.7). A Banach algebra $\mathcal{A}$ is biflat, if there exists a bounded $\mathcal{A}$-bimodule morphism $\rho : \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ such that

$$\pi^{**} \circ \rho(a) = a, \qquad (a \in \mathcal{A}) \tag{1.5}$$

is the canonical embedding of $\mathcal{A}$ into $\mathcal{A}^{**}$, see [18]. For example let G be an amenable locally compact group and S be a left zero semigroup (i.e $st = s$ for each $s, t \in S$). Then by Johnson's theorem $L^1(G)$ is amenable. So $L^1(G)$ is biflat, see [9, Proposition 4.5]. One can easily see that $l^1(S)$ is biflat and so does $L^1(G) \otimes_w l^1(S)$, see [12, Proposition 2.5]. For more informations about biflatness, see [8].

Definition 1. *Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a unital dual Banach algebra and $\varphi \in \Delta_{w^*}(\mathcal{A})$. Suppose that the short exact sequence*

$$\Theta : 0 \to \mathcal{A}_* \underset{\tau}{\overset{\pi^*}{\rightleftharpoons}} \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \frac{\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*}{\pi^*(\mathcal{A}_*)} \to 0 \tag{1.6}$$

of $\mathcal{A}$-bimodules is admissible. Then we say that in (3) sequence Θ, φ-splits if there exists a bounded linear map $\tau : \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \mathcal{A}_*$ such that

$$\tau \circ \pi^*(\varphi) = \varphi, \qquad \tau(\Omega.a) = \varphi(a)\tau(\Omega)$$

for every $a \in \mathcal{A}$ and $\Omega \in \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$, see [10]. Note that $\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$ is a closed submodule of $(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$.

Definition 2. *Let $\mathcal{A}$ be a dual Banach algebra, and $\varphi \in \Delta_{w^*}(\mathcal{A})$. An element $M \in \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**}$ is called a φ-σwc virtual diagonal for $\mathcal{A}$ if*

$$a.M = \varphi(a)M, \qquad \langle \varphi \otimes \varphi, M \rangle = 1 \tag{1.7}$$

for every $a \in \mathcal{A}$, see [10, Definition 3.1].

Note that the normal virtual diagonal [14, Definition 4.4.15] and the φ-σwc virtual diagonal are equivalent, (see [3, 4, 13, 15]).

Now by above notations, for two unital dual Banach algebras $\mathcal{A} = (\mathcal{A}_*)^*$ and $\mathcal{B} = (\mathcal{B}_*)^*$ we consider and apply certain three short exact sequences, which have three non-zero terms as follow;

$$\Theta_\varphi : 0 \to \mathcal{A}_* \underset{\tau_{\mathcal{A}}}{\overset{\pi^*_{\mathcal{A}}}{\rightleftharpoons}} \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \frac{\sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*}{\pi^*_{\mathcal{A}}(\mathcal{A}_*)} \to 0 \tag{1.8}$$

$$\Theta_\psi : 0 \to \mathcal{B}_* \underset{\tau_{\mathcal{B}}}{\overset{\pi^*_{\mathcal{B}}}{\rightleftharpoons}} \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^* \to \frac{\sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^*}{\pi^*_{\mathcal{B}}(\mathcal{B}_*)} \to 0 \tag{1.9}$$

In this paper, we present some characterizations for recent mentioned notions. In fact, the relation between the notions of φ-splitting of the short exact sequences and φ-σwc virtual diagonals of the Bnach algebras, are given. Also, we present some examples in this direction.

2 main results

The aim of this section is to investigate of the relationship between the notions of φ-σwc virtual diagonals of dual Banach algebras and φ-splitting of the related short exact sequences. Also, we investigate some hereditary properties of mentioned notions.

Remark 2.1. Note that in the sequel for the sake brevity, for instance instead the sentence "the short exact sequence Θ_φ, φ-splits" we say "Θ_φ, φ-splits".

In the following, we investigate an equivalence relationship between concepts of φ-σwc virtual diagonal and φ-splitting of the short exact sequence of dual Banach algebras.

Theorem 2.2. *Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then if Θ_φ, φ-splits. Then the Banach algebra $\mathcal{A}$ has a φ-σwc virtual diagonal.*

Proof. Suppose that Θ_φ, φ-splits. Then, there exists a linear map $\tau : \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \mathcal{A}_*$ such that

$$\tau \circ \pi^*(\varphi) = \varphi, \qquad \tau(\Omega.a) = \varphi(a)\tau(\Omega) \tag{2.1}$$

for every $a \in \mathcal{A}, \varphi \in \Delta_{w^*}(\mathcal{A})$ and $\Omega \in \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*$. We show that there exists a φ-σwc virtual diagonal for $\mathcal{A}$. Without loss of the generality, suppose that $\mathcal{A}$ is unital, say $e_{_\mathcal{A}}$. Then $\varphi(e_{_\mathcal{A}}) = 1$. From (2.1), we have

$$\begin{aligned} \langle \varphi \otimes \varphi, \tau^*(e_{_\mathcal{A}})\rangle &= \langle \pi^*(\varphi), \tau^*(e_{_\mathcal{A}})\rangle \\ &= \varphi(e_{_\mathcal{A}}) = 1 \end{aligned}$$

Also, by (2.1) we have

$$\begin{aligned} \langle a.\tau^*(e_{_\mathcal{A}}), \Omega\rangle &= \langle e_{_\mathcal{A}}, \tau(\Omega.a)\rangle \\ &= \langle e_{_\mathcal{A}}, \varphi(a)\tau(\Omega)\rangle \\ &= \varphi(a)\langle \tau^*(e_{_\mathcal{A}}), \Omega\rangle \end{aligned} \tag{2.2}$$

for all $a \in \mathcal{A}$. Now we see that $\tau^*(e_{_\mathcal{A}})$ is desirable. □

Example 2.3. Let $\mathcal{A}$ be an amenable C^*-algebra and let $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then $\mathcal{A}^{**}$ is φ-Connes amenable. Hence, Θ_φ, φ-splits. Then $\mathcal{A}$ has a φ-σwc virtual diagonal. The von Neumann algebras are hold in Theorem 2.2.

Before giving an example let us that remind of the following note.

Remark 2.4. Let $\mathcal{A}$ be a Banach algebra, $(I, \leq)$ be a totally ordered set and let $\Delta(\mathcal{A})$ denotes the character space of all continuous linear functionals on Banach algebra $\mathcal{A}$. By $UP_{_I}(\mathcal{A})$ we denote the set of $I \times I$ upper triangular matrices which its entries belong to $\mathcal{A}$ and we set

$$\| (a_{i,j})_{i,j\in I} \| = \sum_{i,j\in I} \| a_{i,j} \| < \infty.$$

where $a_{i,j}$ are the entries of $\mathcal{A}$ for every $i, j \in I$. With matrix actions and $\| \, . \, \|$ as a norm, $UP_{_I}(\mathcal{A})$ turns into a Banach algebra. These algebras are similar (in properties) to the l^1-Munn algebras that thiers Connes amenability is studied in [7]. As above let $\mathcal{A}$ be a Banach algebra with a right identity that $\Delta(\mathcal{A}) \neq \emptyset$ and also let $(I, \leq)$ has smallest element. Then $UP_{_I}(\mathcal{A})$ is biflat if and only if $\mathcal{A}$ is biflat and I is singleton, see [16, Theorem 5.1].

Corollary 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras, $\psi \in \Delta_{w^*}(\mathcal{B})$ and $\varphi \in \Delta_{w^*}(\mathcal{A})$. Then if $\Theta_{\varphi\otimes\varphi}$, $\varphi \otimes \psi$-splits. Then the Banach algebra $\mathcal{A}\widehat{\otimes}\mathcal{B}$ has a $\varphi \otimes \psi$-σwc virtual diagonal.*

The next example confirms the equivalence of conditions in Corollary 2.5.

Example 2.6. Let X be a compact space. Considering $\mathcal{A} = \begin{pmatrix} 0 & 0 \\ C(X) & C(X) \end{pmatrix}$ equipped with usual matrix multiplication and with respect to the norm topology. Let $\mathcal{A}$ be a biflat Banach algebra, $I = \{1, 2\}$ be singleton and $\mathcal{B} = UP_{_I}(\mathcal{A})$. Define $\varphi : \mathcal{A} \to \mathbb{C}$ and $\psi : \mathcal{B} \to \mathbb{C}$, via $\varphi\begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} = \int_X (f + g)(x)dx$ for all $f, g \in C(X)$ and $\psi(\mathbb{A}) = \sum_{i,j\in I} \| a_{i,j} \| < \infty$ for all $i, j \in I = \{1, 2\}, a_{i,j} \in \mathcal{A}$ and $\mathbb{A} \in UP_{_I}(\mathcal{A})$, respectively. One can see that φ and ψ are linear. Now, put $\varphi \otimes \psi\Big[\begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \otimes \mathbb{A}\Big] = \varphi\begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix}\psi(\mathbb{A})$. We claim that φ is w^*-continuous linear functionals. For this reason, suppose that $U = \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix}, U_\alpha = \begin{pmatrix} 0 & 0 \\ f_\alpha & g_\alpha \end{pmatrix} \in \mathcal{A}$ and

$U_\alpha \overset{w^*-topology}{\longrightarrow} U(\alpha \in I)$. Since ranges of φ and ψ are finite-dimensional and we known that in such spaces all topologies are coincide. It is clear that $g_\alpha \overset{w^*-topology}{\longrightarrow} g$ and $f_\alpha \overset{w^*-topology}{\longrightarrow} f$. Therefore φ is w^*-continuous. As the same way ψ is norm continuous and w^*-continuous. Then by Lemma 2.1, $\varphi \otimes \psi \in \Delta_{w^*}(\mathcal{A}\widehat{\otimes}\mathcal{B})$. Now, define

$$\Big[\begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix} \otimes \mathbb{A}\Big]\Big[\begin{pmatrix} 0 & 0 \\ f' & g' \end{pmatrix} \otimes \mathbb{B}\Big] = \begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix}\begin{pmatrix} 0 & 0 \\ f' & g' \end{pmatrix} \otimes \mathbb{AB} = \begin{pmatrix} 0 & 0 \\ gf' & gg' \end{pmatrix} \otimes \mathbb{AB}$$

for every $f, g \in C(X)$ and $\mathbb{A}, \mathbb{B} \in \mathcal{B}$. Then by [**?**, Chapter 6], $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is a biflat Banach algebra. We can see that $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is $\varphi \otimes \psi$-Connes amenable and thus has a $\varphi \otimes \psi$-σwc virtual diagonal.

Corollary 2.7. *Suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ are unital dual Banach algebras. Suppose that $\varphi \in \Delta_{w^*}(\mathcal{A})$ and $\psi \in \Delta_{w^*}(\mathcal{B})$. If $\mathcal{A}\widehat{\otimes}\mathcal{B}$ has a $\varphi \otimes \psi$-σwc virtual diagonal. Then $\mathcal{A}$ has a φ-σwc virtual diagonal and $\mathcal{B}$ has a ψ-σwc virtual diagonal.*

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