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Some Results on φ-Connes Amenability of Dual Banach Algebras

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Abstract

Let $\mathbb{A}$ be a Banach algebra and $\varphi^\sharp$ be the unique extension of the non-zero functional φ on unitization of $\mathbb{A}$, i. e., $\mathbb{A}^\sharp$, where $\varphi \in \mathbb{A}_*$. In this paper, we present a characterization for character Connes amenability of the second dual of Banach algebra $\mathbb{A}$ by using a homomorphism $\theta : \mathbb{A} \to \mathbb{A}^{**}$ with w^*-dense range. Also, the relation among $\ker \varphi$, $\ker \varphi^\sharp$ and left identities of Banach algebras $\mathbb{A}$ and $\mathbb{A}^\sharp$ is investigated. We generalize this concept to the projective tensor product of unital dual Banach algebras. Some results are also given.

1 Introduction

In [9, 10], Kaniuth et al. introduced the generalization of concept of amenability which depends on non-zero functionals and also, in [11] the mentioned concept is studied by Monfared. In this work we write $\mathbb{A} = (\mathbb{A}_*)^*$ if we wish to emphasize that $\mathbb{A}$ is a dual Banach algebra with predual $\mathbb{A}_*$. A proper concept of amenability for dual Banach algebras is the Connes amenability. This notion under different name, for the first time was introduced by Johnson, Kadison, and Ringrose for von Neumann algebras [8]. Also the concept of Connes amenability for the larger class of dual Banach algebras, which seems to be a natural variant of amenability for dual Banach algebras, systematically was introduced and later extended by Runde [12]. Recently, Ghaffari et al. studied the concept of Connes amenability for l^1-Munn algebras and for certain product of Banach algebras [3, 4]. A dual Banach $\mathbb{A}$-bimodule $\mathbb{E}$ is called normal, if for each $x \in \mathbb{E}$ both of the module maps

$$\mathbb{A} \to \mathbb{E}, \qquad a \mapsto \begin{cases} x.a, \\ a.x \end{cases} \tag{1.1}$$

are w^*-continuous. For a given dual Banach algebra $\mathbb{A}$ and a Banach $\mathbb{A}$-bimodule $\mathbb{E}$, let $\sigma wc(\mathbb{E})$ denotes the set of all elements $x \in \mathbb{E}$ such that the module maps in (2.1), are w^*-continuous. One can see that, $\sigma wc(\mathbb{E})$ is a closed submodule of $\mathbb{E}$. Connes amenability of dual Banach algebra $\mathbb{A}$ is investigated through existence the σwc-virtual diagonals for $\mathbb{A}$ [13, Theorem 4.8]. Also, φ-Connes amenability and φ-invariant mean (φ is a non-zero continuous functional on a Banach algebra) which seem to be natural variants of Connes amenability and Connes mean for dual Banach algebras, systematically were introduced [5]. Another version that is called φ-Connes module amenability of dual Banach algebras and semigroup algebras, is investigated by the authors in [6, 15].

The purpose of the present work is to generalize the notions of Connes amenability by investigating and studying kernel of non-zero functionals on Banach algebras. We summarize outline the contents of present paper. Let $\mathbb{A}$ and $\mathbb{B}$ be dual Banach algebras. Let $\varphi \in \Delta_{w^*}(\mathbb{A})$, be the set of all w^*-continuous non-zero functionals from $\mathbb{A}$ onto a complex plane $\mathbb{C}$, $\psi \in \Delta_{w^*}(\mathbb{B})$ and $\mathbb{E}$ be a Banach $\mathbb{A}$-bimodule. After introducing some notations, Theorem 2.4 gives a characterization for φ-Connes amenability of the second dual of $\mathbb{A}$. In Theorem 2.6, we mainly focus on $\ker \varphi \subseteq \mathbb{A}$ and $\ker \varphi^\sharp \subseteq \mathbb{A}^\sharp$. The relation between φ-invariant mean and $\varphi^\sharp$-invariant mean of Banach algebras $\mathbb{A}$ and $\mathbb{A}^\sharp$ is

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investigated in Corollary 2.8. The main result of this paper (Theorem 2.10) gives the cohomology properties between Banach algebras and their projective tensor product. Indeed, it has been showed that there exists a closely relation among left identities of $\ker\varphi$, $\ker\psi$ and $\ker(\varphi\otimes\psi)$. In Corollary 2.11, we specify $\varphi\otimes\psi$-Connes amenability of $\mathbb{A}\widehat{\otimes}\mathbb{B}$ through existence the left identity for $\ker(\varphi\otimes\psi)$. Finally, in both of the Poroposition 2.12 and Theorem 2.13, under the certain conditions we present illustrative proof of importance of communication the φ-invariant means for Banach algebras.

2 Main results

Let $\mathbb{A}$ be an arbitrary Banach algebra and $\varphi:\mathbb{A}\to\mathbb{C}$ be a non-zero functional. First, we present the fundamental definitions of φ-invariant mean and φ-Connes amenability for a dual Banach algebra $\mathbb{A}$ which will be studied in this paper.

$\mathbb{A}$ is called φ-amenable if there exists a bounded linear functional $\mu\in\mathbb{A}^{**}$ satisfying $\mu(\varphi)=1$ and $\mu(f.a)=\varphi(a)\mu(f)$ for all $a\in\mathbb{A}$ and $f\in\mathbb{A}^*$ [10].

Definition 2.1 *Let $\mathbb{A}=(\mathbb{A}_*)^*$ be a dual Banach algebra and $\varphi\in\Delta_{w^*}(\mathbb{A})\cap\mathbb{A}_*$. We say that a linear functional $\mu\in\mathbb{A}^{**}$ is a mean if $\mu(\varphi)=1$. Also, μ is called φ-invariant mean for $\mathbb{A}$ if*

$$\begin{cases}\mu(\varphi)=1,\\ \mu(f.a)=\varphi(a)\mu(f)\end{cases} \tag{2.1}$$

for all $a\in\mathbb{A}$ and $f\in\mathbb{A}_*$, see [5].

Definition 2.2 *Let $\mathbb{A}$ be a Banch algebra and $\mathbb{E}$ be a dual Banach $\mathbb{A}$-bimodule. A bounded linear map $\mathcal{D}:\mathbb{A}\to\mathbb{E}$ is called a derivation if it satisfies*

$$\mathcal{D}(ab)=\mathcal{D}(a).b+a.\mathcal{D}(b)$$

for all $a,b\in\mathbb{A}$.

Given $x\in\mathbb{E}$, the inner derivation $ad_x:\mathbb{A}\to\mathbb{E}$ is defined by $ad_x(a)=a.x-x.a$.

Definition 2.3 *A dual Banach algebra $\mathbb{A}$ is called φ-Connes amenable if for every normal $\mathbb{A}$-bimodule $\mathbb{E}$, every bounded w^*-continuous derivation $\mathcal{D}:\mathbb{A}\to\mathbb{E}$, is inner, that is for all $a\in\mathbb{A}$ there exists some $x\in\mathbb{E}$ such that $\mathcal{D}(a)=a.x-x.a$, see [14].*

Theorem 2.4 *Let $\mathbb{A}$ be a dual Arens regular Banach algebra, $\Theta:\mathbb{A}\to\mathbb{A}^{**}$ be a continuous and w^*-continuous non-zero functional with w^*-dense range. If $\Phi\in\Delta_{w^*}(\mathbb{A}^{**})$ and $\mathbb{A}$ is $\Phi\circ\Theta$-amenable, then $\mathbb{A}^{**}$ is Φ-Connes amenable.*
Proof. First, consider Figure 1 below:

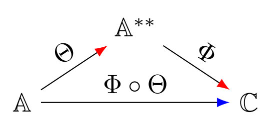

Figure 1: Commutative diagram.

Suppose that $\mathbb{A}$ is a dual Arens regular Banach algebra. Without loss of generality suppose that, $\mathbb{A}$ is an ideal in $\mathbb{A}^{**}$. Using [14, Example 4.4.2(e)], it is clear that $\mathbb{A}^{**}$ whit the first Arens product is a dual Banach algebra. Note that the non-zero functional $\Phi\circ\Theta:\mathbb{A}\to\mathbb{C}$ is w^*-continuous. By assumption and Remark 2.7, it follows that $\mathbb{A}$ is $\Phi\circ\Theta$-Connes amenable. Therefore, there exists μ in $\sigma wc(\mathbb{A}^*)^*$ such that

$$\begin{cases}\mu(\Phi\circ\Theta)=1,\\ \mu(f.a)=\Phi\circ\Theta(a)\mu(f)\end{cases} \tag{2.2}$$

for all $a\in\mathbb{A}$ and $f\in\sigma wc(\mathbb{A}^*)$. Define $\rho\in\sigma wc((\mathbb{A}^{**})^*)^*$ by $\rho(\Gamma):=\mu(\Gamma\circ\Theta)$ for $\Gamma\in\sigma wc(\mathbb{A}^{**})^*$. We have

$$\rho(\Gamma.\Theta(a))=\mu((\Gamma.\Theta(a))\circ\Theta) \tag{2.3}$$

for $a \in \mathbb{A}$ *and* $\Gamma \in \sigma wc(\mathbb{A}^{**})^*$. *We obtain*

$$\begin{aligned}\langle(\Gamma\circ\Theta).a, b\rangle &= \langle(\Gamma\circ\Theta), ab\rangle\\ &= \langle\Gamma, \Theta(a)\Theta(b)\rangle\\ &= \langle(\Gamma.\Theta(a))\circ\Theta, b\rangle\end{aligned}$$

for all $a, b \in \mathbb{A}$.

So,

$$(\Gamma.\Theta(a))\circ\Theta = (\Gamma\circ\Theta).a.$$

Hence by using (2.3),

$$\begin{aligned}\rho(\Gamma.\Theta(a)) &= \mu((\Gamma\circ\Theta).a)\\ &= (\Phi\circ\Theta)(a)\mu(\Gamma\circ\Theta)\\ &= (\Phi\circ\Theta)(a)\rho(\Gamma)\end{aligned}$$

Since $\overline{\Theta(\mathbb{A})}^{w^*} = \mathbb{A}^{**}$, *then* $\mathbb{A}^{**}$ *is* Φ*-Connes amenable.* □

Remark 2.5 *Note that for a dual Banach algebra* $\mathbb{A} = (\mathbb{A}_*)^*$ *and non-zero functional* $\varphi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$, *the existance of left identity for* $\ker\varphi$ *and* φ*-Connes amenability of* $\mathbb{A}$ *are equivalent [5, Proposition 3.6]. Also, it follows that* $\mathbb{A}$ *has a* φ*-invariant mean.*

Theorem 2.6 *Let* $\mathbb{A} = (\mathbb{A}_*)^*$ *be a Banach algebra and* $\varphi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$. *Let* $\varphi^\sharp$, *be the unique extension of* φ *to an element of* $\Delta(\mathbb{A}^\sharp)$. *Then* $\ker\varphi^\sharp$ *has a left identity if and only if* $\ker\varphi$ *has a left identity.*

Proof. First, consider that $\ker\varphi$ *has a left identity, we say* $e_{\mathbb{A}}$. *It follows that* $\mathbb{A}$ *is* φ*-Connes amenable. Thus let* $e_{\mathbb{A}}$ *be the unit of* $\mathbb{A}$, *then the unique extension of* $\mathbb{A}$, $\mathbb{A}^\sharp = \mathbb{A}\oplus\mathbb{C}e_{\mathbb{A}}$, *is* $\varphi^\sharp$*-Connes amenable [5, Proposition 3.6]. Thus, by using [5, Proposition 2.8],* $\ker\varphi^\sharp$ *has a left identity.*

Conversely, suppose that $\ker\varphi^\sharp$ *has a left identity and* $\mathbb{E}$ *be a normal* $\mathbb{A}$*-bimodule. Suppose that* $\mathcal{D} : \mathbb{A} \to \mathbb{E}$ *is a bounded* w^**-continuous derivation. To see that* $\mathcal{D}$ *is inner, consider* $x \in \mathbb{E}$ *and define* $x.e = e.x = x$. *It is clear that* $\mathbb{E}$ *is a normal* $\mathbb{A}^\sharp$*-bimodule. For extend* $\mathcal{D}$, *we define* $\mathcal{D}^\sharp : \mathbb{A}^\sharp \to \mathbb{E}$ *by setting*

$$\mathcal{D}^\sharp(a + \xi e_{\mathbb{A}}) = \mathcal{D}(a),$$

for $a \in \mathbb{A}$ *and* $\xi \in \mathbb{C}$. *It is easy to see that* $\mathcal{D}^\sharp$ *is* w^**-continuous derivation. By hypothesis, it follows that* $\mathbb{A}^\sharp$ *is* $\varphi^\sharp$*-Connes amenable.* $\mathcal{D}^\sharp$ *is thus inner, and so is* $\mathcal{D}$. □

Remark 2.7 *Suppose that* $\mathbb{A}$ *is a dual Banach algebra and* $\varphi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$. *Clearly, every* φ*-amenable Banach algebra is* φ*-Connes amenable. Also, it is shown that a dual Banach algebra* $\mathbb{A}$ *is* φ*-Connes amenable, if and only if* $\mathbb{A}^{**}$ *has a* φ*-invariant mean on* $\mathbb{A}_*$.

Corollary 2.8 *Let* $\mathbb{A} = (\mathbb{A}_*)^*$ *be a Banach algebra and* $\varphi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ *and* $\varphi^\sharp \in \Delta_{w^*}(\mathbb{A}^\sharp) \cap \mathbb{A}^\sharp{}_*$. *Then* $\mathbb{A}$ *has a* φ*-invariant mean if and only if* $\mathbb{A}^\sharp$ *has a* $\varphi^\sharp$*-invariant mean.*

Proof. By an argument similar to the proof of Theorem 2.6, able to completes the proof. □

Definition 2.9 *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two dual Banach algebras,* $\varphi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ *and* $\psi \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$. *We say that* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *is* $\varphi\otimes\psi$*-Connes amenable if for every normal* $\mathbb{A}\widehat{\otimes}\mathbb{B}$*-bimodule* $\mathbb{E}$, *every bounded* w^**-continuous derivation from* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *to* $\mathbb{E}$, *is inner.*

Our next goal is thus to study the relation between left identities of $\ker\varphi$, $\ker\psi$ and $\ker(\varphi\otimes\psi)$.

Theorem 2.10 *Let* $\mathbb{A} = (\mathbb{A}_*)^*$, $\mathbb{B} = (\mathbb{B}_*)^*$ *and* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *be unital dual Banach algebras. Suppose that* $\varphi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ *and* $\psi \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$. *Then the following two conditions are equivalent:*

1. $\ker\varphi$ *and* $\ker\psi$ *have left identities;*
2. $\ker(\varphi\otimes\psi)$ *has a left identity.*

Proof. (1) $\Rightarrow$ (2): *Suppose that both of* $\ker\varphi$ *and* $\ker\psi$ *have left identities. Then by Remark 2.7,* $\mathbb{A}$ *is* φ*-Connes amenable and* $\mathbb{B}$ *is* ψ*-Connes amenable. We claim that the second dual of* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *has a* $\varphi\otimes\psi$*-invariant mean on predual of* $\mathbb{A}\widehat{\otimes}\mathbb{B}$*. By hypothesis, let* $e_{\mathbb{A}}, e_{\mathbb{B}}$ *and* $e_{\mathbb{A}\widehat{\otimes}\mathbb{B}}$ *be the units of* $\mathbb{A}$*,* $\mathbb{B}$ *and* $\mathbb{A}\widehat{\otimes}\mathbb{B}$*, respectively. We make* $\mathbb{A}$ *into a normal* $\mathbb{A}$*-bimodule and* $\mathbb{B}$ *into a normal* $\mathbb{B}$*-bimodule through module actions as follows*

$$a.x = \varphi(a)x \qquad c.z = \psi(c)z \qquad (x, a \in \mathbb{A}, \quad z, c \in \mathbb{B}). \tag{2.4}$$

We define the following bilinear map

$$\kappa : \mathbb{A}\times\mathbb{B} \to \mathbb{C}, \qquad \kappa(a,b) = \langle a, \mathbf{h}\rangle\langle b, \mathbf{f}\rangle.$$

for each $\mathbf{h} \in \mathbb{A}_*$ *and* $\mathbf{f} \in \mathbb{B}_*$*. By using [1, Theorem 6], we can find an unique mapping* $\Omega_\kappa : \mathbb{A}\widehat{\otimes}\mathbb{B} \to \mathbb{C}$ *such that* $\Omega_\kappa(a\otimes b) = \kappa(a,b)$*. Let* $\{a_i\} \subseteq \mathbb{A}$ *and* $\{b_j\} \subseteq \mathbb{B}$ *be two nets such that*

$$w^* - \lim_i a_i = \mathbf{H} \in \mathbb{A}^{**} \tag{2.5}$$

and

$$w^* - \lim_j b_j = \mathbf{F} \in \mathbb{B}^{**}. \tag{2.6}$$

In this case, by (2.5) and (2.6) we can extend κ *to a following continuous mapping*

$$\overline{\kappa} : \mathbb{A}^{**}\times\mathbb{B}^{**} \to \mathbb{C}, \qquad \overline{\kappa}(\mathbf{H},\mathbf{F}) = w^* - \lim_i (w^* - \lim_j \kappa(a_i, b_j)). \tag{2.7}$$

By [7, Lemma 1.7], there exists a linear mapping such as

$$\Theta : \mathbb{A}^{**}\widehat{\otimes}\mathbb{B}^{**} \to (\mathbb{A}\widehat{\otimes}\mathbb{B})^{**}, \qquad \langle\Theta(\mathbf{H}\otimes\mathbf{F}), \Omega_\kappa\rangle = \overline{\kappa}(\mathbf{H},\mathbf{F}) \tag{2.8}$$

that is continuous. Let μ_1 *be a* φ*-invariant mean on* $\mathbb{A}_*$ *and* μ_2 *be a* ψ*-invariant mean on* $\mathbb{B}_*$*. Now, consider nets* $\{a_i\} \subseteq \mathbb{A}$ *and* $\{b_j\} \subseteq \mathbb{B}$ *such that* $a_i \stackrel{w^*}{\to} \mu_1$ *and* $b_j \stackrel{w^*}{\to} \mu_2$ *on the second duals of* $\mathbb{A}$ *and* $\mathbb{B}$*, respectively. By using (2.4), (2.8) and (2.7), we have*

$$\begin{aligned}
\langle\Theta(\mu_1\otimes\mu_2), (a\otimes b).(\mathbf{h}\otimes\mathbf{f})\rangle &= \langle\Theta(\mu_1\otimes\mu_2), (a.\mathbf{h}\otimes b.\mathbf{f})\rangle \\
&= w^* - \lim_i w^* - \lim_j a.\mathbf{h}\otimes b.\mathbf{f}\langle a_i, b_j\rangle \\
&= w^* - \lim_i w^* - \lim_j \langle a.\mathbf{h}, a_i\rangle\langle b.\mathbf{f}, b_j\rangle \\
&= \langle a.\mathbf{h}, \mu_1\rangle\langle b.\mathbf{f}, \mu_2\rangle \\
&= \varphi(a)\langle\mathbf{h}, \mu_1\rangle\psi(b)\langle\mathbf{f}, \mu_2\rangle \\
&= (\varphi\otimes\psi)(a\otimes b)\langle\mathbf{h}, \mu_1\rangle\langle\mathbf{f}, \mu_2\rangle \\
&= (\varphi\otimes\psi)(a\otimes b)\langle\Theta(\mu_1\otimes\mu_2), (\mathbf{h}\otimes\mathbf{f})\rangle
\end{aligned}$$

for all $a\otimes b \in \mathbb{A}\widehat{\otimes}\mathbb{B}$*. Therefore* $\Theta(\mu_1\otimes\mu_2)$ *is a* $\varphi\otimes\psi$*-invariant mean. By Remark 2.7,* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *is* $\varphi\otimes\psi$*-Connes amenable. Therefore by Remark 2.5, the proof is completes.*

(2) $\Rightarrow$ (1): *This part follows as in the proof of [10, Theorem 3.3].* □

As an straightforward consequence of Theorem 2.10, we have the next result.

Corollary 2.11 *With above notations,* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *is* $\varphi\otimes\psi$*-Connes amenable if and only if* $\ker(\varphi\otimes\psi)$ *has a left identity.*

Proof. First, suppose that $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *is* $\varphi\otimes\psi$*-Connes amenable. We denote the unit of* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *with* $e_{\mathbb{A}\widehat{\otimes}\mathbb{B}}$*. We make* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *into a normal* $\mathbb{A}\widehat{\otimes}\mathbb{B}$*-bimodule via the module actions*

$$(c\otimes d).x = \varphi\otimes\psi(c\otimes d)x, \qquad x.(c\otimes d) = x(c\otimes d)$$

for all $c \in \mathbb{A}, d \in \mathbb{B}, x \in \mathbb{A}\widehat{\otimes}\mathbb{B}$*. We define*

$$\mathcal{D} : \mathbb{A}\widehat{\otimes}\mathbb{B} \to \mathbb{A}\widehat{\otimes}\mathbb{B}, \quad c\otimes d \longmapsto c\otimes d - \varphi\otimes\psi(c\otimes d)e_{\mathbb{A}\widehat{\otimes}\mathbb{B}} \tag{2.9}$$

It is clear that $\mathcal{D}$ is bounded w^*-continuous derivation. Also, $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is mapping by $\mathcal{D}$ into the submodule $\ker(\varphi\otimes\psi)\subseteq \mathbb{A}\widehat{\otimes}\mathbb{B}$ which is closed in the w^*-topology. So, by hypothesis we can find element $y\in\ker(\varphi\otimes\psi)$ with following property

$$\mathcal{D}(c\otimes d)=ad_y(c\otimes d)=(c\otimes d).y-y.(c\otimes d) \tag{2.10}$$

for every $c\in\mathbb{A}, d\in\mathbb{B}$. Let $(c\otimes d)\in\ker(\varphi\otimes\psi)$, by (2.9) and (2.10), we have

$$(c\otimes d).y-y.(c\otimes d)=(c\otimes d)-\varphi\otimes\psi(c\otimes d)e_{\mathbb{A}\widehat{\otimes}\mathbb{B}}$$

and so $(c\otimes d)=-y(c\otimes d)$. This argument shows that $\ker(\varphi\otimes\psi)$ has a left identity.

The converse follows easily by the same argument as in the proof of Theorem 2.10. □

Let $\mathbb{A}$ be a dual Banach algebra and $\mathbb{E}$ be a Banach $\mathbb{A}$-bimodule. We write $\mathcal{Z}^1_{w^*}(\mathbb{A},\mathbb{E})$ and $\mathcal{N}^1_{w^*}(\mathbb{A},\mathbb{E})$, for the space of all w^*-derivations and the inner w^*-derivations from $\mathbb{A}$ onto $\mathbb{E}$, respectively. We recall that the quotient space

$$\mathcal{H}^1_{w^*}(\mathbb{A},\mathbb{E})=\frac{\mathcal{Z}^1_{w^*}(\mathbb{A},\mathbb{E})}{\mathcal{N}^1_{w^*}(\mathbb{A},\mathbb{E})}$$

is the first Hochschild cohomology group of $\mathbb{A}$ with coefficients in $\mathbb{E}$ with respect to w^*-topology.

It is known that a Banach algebra $\mathbb{A}$ is amenable if $\mathcal{H}^1_{w^*}(\mathbb{A},\mathbb{E}^*)=0$, for every dual Banach $\mathbb{A}$-bimodule $\mathbb{E}^*$ with a canonical action.

Proposition 2.12 *Suppose that $\mathbb{A}$ is a dual Banach algebra and $\varphi\in\Delta_{w^*}(\mathbb{A})\cap\mathbb{A}_*$. Then for each Banach $\mathbb{A}$-bimodule $\mathbb{E}$, the following three conditions are equivalent:*

1. *$\mathbb{A}$ has a φ-invariant mean;*
2. *$\mathcal{H}^1_{w^*}(\mathbb{A},\mathbb{E}^{**})=\{0\}$;*
3. *for each w^*-continuous derivation $\mathcal{D}$ from $\mathbb{A}$ to $\mathbb{E}$, there exists a bounded net (x_i) in $\mathbb{E}$ by property*

$$\mathcal{D}a=w^*-lim_i(a.x_i-x_i.a),\qquad (a\in\mathbb{A}).$$

Proof. $(1)\Rightarrow(2)$*: It follows immediately from definitions.*
$(2)\Rightarrow(3)$*: Consider Figure 2 below:*

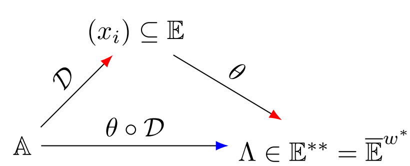

Figure 2: Commutative diagram.

Let for each Banach $\mathbb{A}$-bimodule $\mathbb{E}$, $\mathcal{H}^1_{w^*}(\mathbb{A},\mathbb{E}^{**})=0$. Suppose that $\mathcal{D}\in\mathcal{Z}^1_{w^*}(\mathbb{A},\mathbb{E})$, $\theta:\mathbb{E}\to\mathbb{E}^{**}$ is the canonical embedding. Then $\theta\circ\mathcal{D}\in\mathcal{Z}^1_{w^*}(\mathbb{A},\mathbb{E}^{**})$. By clause (ii), there exists $\Lambda\in\mathbb{E}^{**}$ such that $\theta\circ\mathcal{D}(a)=a.\Lambda-\Lambda.a$, for all $a\in\mathbb{A}$. By Goldstein theorem, $\overline{\mathbb{E}}^{w^*}=\mathbb{E}^{**}$. Thus, there exists a bounded net $(x_i)\subseteq\mathbb{E}$ such that $x_i\overset{w^*}{\to}\Lambda$. In view of [2, Theorem A.3.29, (i) and (ii)], and by applying Goldstein and Mazur theorems, we obtain

$$\begin{aligned}\mathcal{D}(a)&=a.(w^*-\lim_i x_i)-(w^*-\lim_i x_i).a\\&=w^*-\lim_i(a.x_i-x_i.a).\end{aligned}$$

So, (x_i) satisfies the requirements in (3).

$(3)\Rightarrow(1)$*: Suppose that $\mathcal{D}\in\mathcal{Z}^1_{w^*}(\mathbb{A},\mathbb{E})$, let $(x_i)\subseteq\mathbb{E}$ be a bounded net such that $\mathcal{D}a=w^*-\lim_i(a.x_i-x_i.a)$. Assume $x_i\overset{w^*}{\to}x$. Therefore $\mathcal{D}a=a.x-x.a$, and then $\mathcal{D}\in\mathcal{N}^1_{w^*}(\mathbb{A},\mathbb{E})$. Thus $\mathbb{A}$ is φ-Connes amenable. According to Remark 2.5, $\mathbb{A}$ has a φ-invariant mean. Realy, the above argument shows that conditions of (1), (2) and (3) are equivalent, as required.* □

By Remark 2.3, it is clear that every φ-amenable Banach algebra is φ-Connes amenable, which φ is a non-zero functional on an arbitrary Banach algebra. The goal of the following theorem is to investigate the relation between the concepts of φ-amenability and φ-Connes amenability of a dual Banach algebra under certain condition.

Theorem 2.13 *Let $\mathbb{A}$ be a Banach algebra, $\mathbb{E}$ be a Banach $\mathbb{A}$-bimodule and $\varphi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$. Then the following three statements are equivalent:*

1. *$\mathbb{A}$ has a φ-invariant mean;*
2. *Under the module action of $\mathbb{A}$ on $\mathbb{E}$ given by $a.x = \varphi(a)x$ for all $x \in \mathbb{E}$ and $a \in \mathbb{A}$, we have $\mathcal{H}^1_{w^*}(\mathbb{A}, \mathbb{E}^*) = \{0\}$;*
3. *For $\mathbb{A}$-bimodule $\sigma wc(\ker \varphi)^{**}$ such that $a.\mathcal{F} = \varphi(a)\mathcal{F}$ for $\mathcal{F} \in \sigma wc(\ker \varphi)^{**}$ and $a \in \mathbb{A}$, each $\mathcal{D} \in \mathcal{Z}^1_{w^*}(\mathbb{A}, \sigma wc(\ker \varphi)^{**})$ is inner.*

Proof. (1) $\Rightarrow$ (2): *Suppose that $\mu \in \mathbb{A}^{**}$ is a φ-invariant mean that satisfies in condition (2.1) and $\mathcal{D} : \mathbb{A} \to \mathbb{E}_*$ is a bounded w^*-continuous derivation. Suppose that $\mathcal{D}^* : \mathbb{E} \to \mathbb{A}^*$ is the strict of adjoint of $\mathcal{D}$ on $\mathbb{E}$. Moreover, consider $\mathcal{D}^{**}(\mu) = \xi \in \mathbb{E}^*$. By using of hypothesis in (2), we have*

$$\langle b, \mathcal{D}^*(a.x)\rangle = \langle a.x, \mathcal{D}(b)\rangle = \varphi(a)\langle b, \mathcal{D}^*(x)\rangle$$

for every $a, b \in \mathbb{A}$ and $x \in \mathbb{E}$. It follows that

$$\mathcal{D}^*(a.x) = \varphi(a)\mathcal{D}^*(x)$$

and

$$\langle x, \xi.a\rangle = \langle \mathcal{D}^*(a.x), \mu\rangle = \varphi(a)\langle x, \xi\rangle.$$

Then

$$\xi.a = \varphi(a)\xi \tag{2.11}$$

for $a \in \mathbb{A}$. As similar argument we obtain

$$\mathcal{D}^*(x.a) = \mathcal{D}^*(x).a - \langle x, \mathcal{D}(a)\rangle\varphi$$

for every $x \in \mathbb{E}$ and $a \in \mathcal{A}$. Therefore

$$\langle x, a.\xi\rangle = \varphi(a)\langle x, \xi\rangle - \langle x, \mathcal{D}(a)\rangle.$$

We conclude that

$$\mathcal{D}(a) = \varphi(a)\xi - a.\xi. \tag{2.12}$$

By (2.11), (2.12) and hypothesis in (2), we have

$$\mathcal{D}(a) = \mathcal{D}_{-\xi}(a)$$

for $a \in \mathcal{A}$ and so, $\mathcal{H}^1_{w^*}(\mathbb{A}, \mathbb{E}^*) = \{0\}$ is hold. Also, it is clear that (2) and (3) are equivalent. Now let (3) holds. We show that (1) is hold. By the properties in clause (3), define a derivation $\mathcal{D}$ from $\mathbb{A}$ into $\sigma wc(\ker \varphi)^{**}$. Since $\mathcal{D}$ is inner derivation, therefore we only have to show $\mathbb{A}$ admits a φ-invariant mean. Choose any $u \in \mathbb{A}$ such that $\varphi(u) = 1$. We have

$$\begin{aligned}(au - ua).b + a.(bu - ub) &= ab.u - u.ab \\ &= \mathcal{D}(ab) \\ &= ad_u(ab)\end{aligned}$$

for all $a, b, u \in \mathbb{A}$, then $\mathcal{D}a = a.u - u.a$. On the other hand, since $\mathcal{D}$ is inner, there exists $\mathcal{F} \in \sigma wc(\ker \varphi)^{**}$ with

$$\mathcal{D}a = a.(-\mathcal{F}) - (-\mathcal{F}).a$$

for all $a \in \mathbb{A}$. So,

$$a.u - u.a = a.(-\mathcal{F}) - (-\mathcal{F}).a.$$

Then by hypothesis of the left module action we have,

$$a.(u+\mathcal{F}) = (u+\mathcal{F}).a = \varphi(a)(u+\mathcal{F})$$

for all $a \in \mathbb{A}$. Now, we have

$$\begin{aligned}(u+\mathcal{F})(f.a) &= \langle u+\mathcal{F}, f.a\rangle \\ &= \langle a.(u+\mathcal{F}), f\rangle \\ &= \varphi(a)\langle u+\mathcal{F}, f\rangle \\ &= \varphi(a)(u+\mathcal{F})(f)\end{aligned}$$

for all $f \in \sigma wc(\mathbb{A}^*)$ and $a \in \mathbb{A}$. Also, $\langle u+\mathcal{F}, \varphi\rangle = \varphi(u) = 1$. Thus, in addation $u+\mathcal{F} \in \sigma wc(\mathbb{A}^{**})$ is a φ-invariant mean. □

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