

A SURVEY ON χ -MODULE CONNES AMENABILITY OF SEMIGROUP ALGEBRAS

E. Tamimi* and A. Ghaffari

ABSTRACT. We shall study the χ -module Connes amenability of a semigroup algebra $l^1(S)$, where χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is w^* -continuous and S is an inverse weakly cancellative semigroup with subsemigroup E of idempotents. We are mainly concerned with the study of χ -module normal, virtual diagonals. We characterize the χ -module Connes amenability of a semigroup algebra $l^1(S)$. Also, we show that if $l^1(S)$ as a Banach module over $l^1(E)$ has an id -module normal, virtual diagonal then it is id -module Connes amenable. Other characterizations of χ -module Connes amenability of $l^1(S)$ is presented.

1. INTRODUCTION

Connes amenability of dual Banach algebras were introduced by Runde in [19]. In [20], Runde showed that if a Banach algebra is Connes amenable, it has a normal, virtual diagonal. In [7], φ -Connes amenability of dual Banach algebras, where φ is a character from a Banach algebra onto \mathbb{C} is investigated. In [1], Amini introduced the notion of module amenability for semigroup algebras. Amini proved that for an inverse semigroup S with subsemigroup E of idempotents, $l^1(S)$ as a Banach module over $l^1(E)$ is module amenable if S is amenable and vice versa. Also, in [8] Ghaffari et al. studied φ -module Connes amenability of dual Banach algebras, where φ is a w^* -continuous bounded module homomorphism from a Banach algebra to itself. For more study on module amenability, we can refer to [1, 2, 17, 21]. In [14], Minapoor introduced ideal Connes amenability of l^1 -Munn algebras and its application to semigroup algebras. Also, ideal Connes amenability of Lau product of Banach algebras is studied by Minapoor et al. [15]. Module operator virtual diagonals on the Fourier algebra of an inverse semigroup is studied by Amini and Rezavand in [3]. In [13], Lau and Zhang studied the fixed point properties of semigroups of non-expansive mappings on weakly compact convex subsets of a Banach space. Also, they provide a characterization for the existence

Published online: 1 April 2024

MSC(2010): Primary: 43A20, 43A10; Secondary: 22D15.

Keywords: χ -module Connes amenable; Module χ -derivation; χ -module normal virtual diagonal; Weakly cancellative semigroup.

Received: 1 April 2023, Accepted: 29 May 2023.

*Corresponding author.

of a left invariant mean on the space of weakly almost periodic functions on separable semitopological semigroups in terms of fixed point property for non-expansive mappings. In [10, Lemma 2.1], Lau showed that if S is a semitopological semigroup and X is a left translation invariant subspace of the closed subalgebra of $l^\infty(S)$ consisting of continuous functions, where X has a left invariant mean, then S is left reversible.

In [11], Lau and Zhang studied the algebraic and analytic properties of semigroups related to fixed point properties of non-expansive mappings. Also, they investigate the extremely left amenability of a semitopological semigroup in terms of existence a left invariant mean on the left uniformly continuous functions. In [12], Lau and Zhang investigated the fixed point properties for semigroups of non-expansive mappings on convex sets in dual Banach spaces. Also, they showed that any left invariant mean on a left invariant subspace of $l^\infty(S)$ that S is a left reversible semitopological semigroup extends to a strictly increasing left subinvariant submean on $l^\infty(S)$.

All of these concepts generalized the earlier concept of amenability for the semigroup algebras introduced by Johnson [9].

In this paper, we introduce the concept of χ -module Connes amenability for semigroup algebra $l^1(S)$ and we characterize the χ -module Connes amenability in terms of χ -modul normal virtual diagonals. In particular, we show that if $\chi : l^1(S) \rightarrow l^1(S)$ is a bounded module homomorphism that is w^* -continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ -module Connes amenable, then it has a χ -module normal virtual diagonal. Other results and hereditary properties in this direction are also obtained.

2. χ -MODULE CONNES AMENABILITY

A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $x \in S$ is called an idempotent if $x = x^* = x^2$. The set of idempotent elements of S is denoted by E . For $s \in S$, we define $L_s, R_s : S \rightarrow S$ by $L_s(t) = st, R_s(t) = ts, (t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative. Before turning our result, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$ (see [6]).

Let S be an inverse semigroup, $s, t \in S$ and E be a subsemigroup of idempotents. We consider an equivalence relation on S where $s \sim t$ if and only if there is $q \in E$ such that $sq = tq$. The quotient semigroup $S_G = \frac{S}{\sim}$ is a group (see [16]). Also, E is a symmetric subsemigroup of S . Thus, $l^1(S)$

is a Banach $l^1(E)$ -module with compatible canonical actions. Let $l^1(E)$ acts on $l^1(S)$ via

$$\delta_q.\delta_s = \delta_s, \quad \delta_s.\delta_q = \delta_{sq} = \delta_s * \delta_q \quad (s \in S, q \in E). \quad (2.1)$$

It is known that, $l^1(S_G)$ is a quotient of $l^1(S)$ and so the above action of $l^1(E)$ on $l^1(S)$ lifts to an action of $l^1(E)$ on $l^1(S_G)$, making it a Banach $l^1(E)$ -module (see [1]).

Throughout this paper, it is assumed that $l^1(S) = (c_0(S))^*$ is a dual Banach algebra, and $\mathcal{U} = l^1(E)$ is a Banach algebra such that $l^1(S)$ is a Banach \mathcal{U} -bimodule via,

$$\alpha.(\delta_s\delta_t) = (\alpha.\delta_s).\delta_t, \quad (\alpha\beta).\delta_s = \alpha.(\beta.\delta_s) \quad (\delta_s, \delta_t \in l^1(S), \alpha, \beta \in \mathcal{U}). \quad (2.2)$$

Definition 2.1. Let S be a discrete semigroup and $l^1(S)$ be a dual Banach algebra. A module homomorphism from $l^1(S)$ to $l^1(S)$ is a map $\chi : l^1(S) \rightarrow l^1(S)$ with

$$\chi(\alpha.\delta_s + \delta_t.\beta) = \alpha.\chi(\delta_s) + \chi(\delta_t).\beta, \quad \chi(\delta_s\delta_t) = \chi(\delta_s)\chi(\delta_t)$$

for every $\delta_s, \delta_t \in l^1(S)$ and $\alpha, \beta \in \mathcal{U}$.

Throughout this paper $\mathcal{H}_{w^*}(l^1(S))$ will denote the space of all bounded module homomorphisms from $l^1(S)$ to $l^1(S)$ that are w^* -continuous.

Definition 2.2. Let S be a discrete semigroup, $l^1(S) = (c_0(S))^*$ be a dual Banach algebra, X be a dual Banach $l^1(S)$ -bimodule and $\chi \in \mathcal{H}_{w^*}(l^1(S))$. A bounded map $D : l^1(S) \rightarrow X$ is called a module χ -derivation if for every $\delta_s, \delta_t \in l^1(S)$ and $\alpha, \beta \in \mathcal{U}$, we have

$$D(\alpha.\delta_s \pm \delta_t.\beta) = \alpha.D(\delta_s) \pm D(\delta_t).\beta \quad (2.3)$$

and

$$D(\delta_s\delta_t) = D(\delta_s).\chi(\delta_t) + \chi(\delta_s).D(\delta_t). \quad (2.4)$$

Also, X is called symmetric, if

$$\alpha.x = x.\alpha \quad (\alpha \in \mathcal{U}, x \in X).$$

When X is symmetric, each $x \in X$ defines a module χ -derivation

$$(D)_x(\delta_s) = \chi(\delta_s).x - x.\chi(\delta_s) \quad (\delta_s \in l^1(S)). \quad (2.5)$$

Derivations of this form is called inner module χ -derivations.

Let X be a dual Banach $l^1(S)$ -bimodule. Then X is called normal if for each $x \in X$, the maps

$$l^1(S) \rightarrow X; \quad \delta_s \rightarrow \delta_s.x, \quad \delta_s \rightarrow x.\delta_s \quad (2.6)$$

are w^* -continuous. Moreover, if X is an \mathcal{U} -bimodule such that for $x \in X$, $\alpha \in \mathcal{U}$ and $\delta_s \in l^1(S)$ we have

$$\alpha.(\delta_s.x) = (\alpha.\delta_s).x, \quad (\delta_s.\alpha).x = \delta_s.(\alpha.x), \quad (\alpha.x).\delta_s = \alpha.(x.\delta_s), \quad (2.7)$$

then X is called a normal Banach left $l^1(S)$ - \mathcal{U} -module. Similarly for the right and two sided actions.

Definition 2.3. Let S be a discrete semigroup, $l^1(S)$ be a dual semigroup algebra, $\chi \in \mathcal{H}_{w^*}(l^1(S))$ and \mathcal{U} be a semigroup algebra such that $l^1(S)$ is a Banach \mathcal{U} -module. Then $l^1(S)$ is called χ -module Connes amenable if for any symmetric normal Banach $l^1(S)$ - \mathcal{U} -module X , each w^* -continuous module χ -derivation $D : l^1(S) \rightarrow X$ is inner.

In the sequel, we characterize the χ -module Connes amenability of semigroup algebras.

Proposition 2.4. *Let S be a discrete semigroup, $l^1(S)$ be a dual semigroup algebra and $\chi \in \mathcal{H}_{w^*}(l^1(S))$ such that $\chi(l^1(S)) = l^1(S)$. If $l^1(S)$ is χ -module Connes amenable, then $l^1(S)$ is module Connes amenable.*

Proof. Let E be a symmetric normal Banach $l^1(S)$ - $l^1(S)$ -module and $D : l^1(S) \rightarrow E$ be a w^* -continuous module derivation. Set $\rho = D \circ \chi$. The mapping $\rho : l^1(S) \rightarrow E$ is a module χ -derivation. Since $\chi \in \mathcal{H}_{w^*}(l^1(S))$, ρ is w^* -continuous. Thus there exists $r \in E$ such that $\rho(\delta_s) = r.\chi(\delta_s) - \chi(\delta_s).r$ for all $\delta_s \in l^1(S)$. By hypothesis, there exists $\delta_t \in l^1(S)$, such that $\chi(\delta_t) = \delta_s$. Hence

$$\begin{aligned} D(\delta_s) &= D(\chi(\delta_t)) = \rho(\delta_t) \\ &= r.\delta_s - \delta_s.r. \end{aligned}$$

It follows that $l^1(S)$ is module Connes amenable. \square

In [5, Theorem 1.3], Dales and Strauss showed that if S is an infinite, weakly cancellative semigroup, then S is strongly Arens irregular. Also, in [5, Theorem 1.6] by similar conditions they showed that $l^1(S)$ is strongly Arens irregular. Using this notions we prove the following theorem.

Theorem 2.5. *Let S be a finite discrete semigroup, $l^1(S)$ be a dual Arens regular semigroup algebra and $\chi \in \mathcal{H}_{w^*}(l^1(S))$. Then $l^1(S)$ is χ -module Connes amenable if and only if $l^1(S)^{**}$ is χ^{**} -module Connes amenable.*

Proof. Consider the following commutative diagram,

Let $l^1(S)$ be χ -module Connes amenable, E be a symmetric normal Banach $l^1(S)^{**}$ - $l^1(E)$ -module and $D : l^1(S)^{**} \rightarrow E$ be a w^* -continuous module

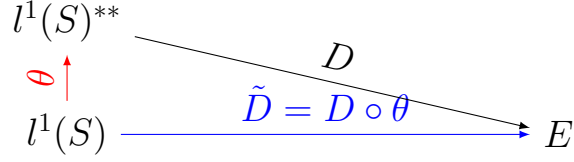


FIGURE 1. Commutative diagram.

χ^{**} -derivation. Let $\theta : l^1(S) \rightarrow l^1(S)^{**}$ be the canonical map. It is clear that θ is w^* -continuous. Define a module action of $l^1(S)$ on E by

$$x \otimes \delta_s = x.\theta(\delta_s), \quad \delta_s \otimes x = \theta(\delta_s).x \quad (\delta_s \in l^1(S), x \in E). \quad (2.8)$$

This module action is well-defined and turns E into a normal Banach $l^1(S)$ - $l^1(E)$ -module. Define a derivation $\tilde{D} : l^1(S) \rightarrow E$ by $\tilde{D} = D \circ \theta$. Then we obtain

$$\begin{aligned} \tilde{D}(\delta_s \delta_t) &= D \circ \theta(\delta_s \delta_t) = D \circ \theta(\delta_s).\chi^{**}(\theta(\delta_t)) + \chi^{**}(\theta(\delta_s)).D \circ \theta(\delta_t) \\ &= D \circ \theta(\delta_s).\theta(\chi(\delta_t)) + \theta(\chi(\delta_s)).D \circ \theta(\delta_t) \\ &= D \circ \theta(\delta_s) \otimes \chi(\delta_t) + \chi(\delta_s) \otimes D \circ \theta(\delta_t) \\ &= \tilde{D}(\delta_s) \otimes \chi(\delta_t) + \chi(\delta_s) \otimes \tilde{D}(\delta_t). \end{aligned}$$

Thus, using the relations (2.3) and (2.4), \tilde{D} is a module χ -derivation that is w^* -continuous. Since $l^1(S)$ is χ -module Connes amenable, there exists $x \in E$ such that

$$\begin{aligned} \tilde{D}(\delta_s) &= D \circ \theta(\delta_s) \\ &= x \otimes \chi(\delta_s) - \chi(\delta_s) \otimes x \\ &= x.\theta(\chi(\delta_s)) - \theta(\chi(\delta_s)).x. \end{aligned} \quad (2.9)$$

Let $\Gamma \in l^1(S)^{**}$. Since $\overline{\theta(l^1(S))}^{w^*} = l^1(S)^{**}$ and using Goldstine Theorem, there exists a net $\{\delta_\alpha\} \subseteq l^1(S)$ such that $\theta(\delta_\alpha) \rightarrow \Gamma$ in the w^* -topology. It can be shown that χ^{**} is w^* -continuous. So, $\chi^{**}(\theta(\delta_\alpha)) \rightarrow \chi^{**}(\Gamma)$. Thus, by relation (2.9)

$$\begin{aligned} D(\Gamma) &= \lim_{\alpha} D \circ \theta(\delta_\alpha) \\ &= \lim_{\alpha} x.\theta \circ \chi(\delta_\alpha) - \theta \circ \chi(\delta_\alpha).x \\ &= \lim_{\alpha} x.\chi^{**} \circ \theta(\delta_\alpha) - \chi^{**} \circ \theta(\delta_\alpha).x \\ &= x.\chi^{**}(\Gamma) - \chi^{**}(\Gamma).x. \end{aligned}$$

Therefore, D is inner.

Conversely, consider the following commutative diagram,

$$\begin{array}{ccc}
 (c_0(S))^* & & \\
 \uparrow \mathbb{E} & \searrow D & \\
 l^1(S)^{**} = (c_0(S))^{***} & \xrightarrow{D \circ \pi} & E
 \end{array}$$

FIGURE 2. Commutative diagram.

Suppose that E is a symmetric normal Banach $l^1(S)$ - $l^1(E)$ -module and $D : l^1(S) \rightarrow E$ is a w^* -continuous module χ -derivation. Let

$$\pi : (c_0(S))^{***} \rightarrow (c_0(S))^*, \quad \pi(\Lambda) = \Lambda |_{\theta(c_0(S))}$$

be the Dixmier projection. Since, π from $l^1(S)^{**}$ onto $l^1(S)$ is a module homomorphism, E is a Banach $l^1(S)^{**}$ - $l^1(E)$ -module with the bimodule actions are defined as follows:

$$\Lambda \ominus x = \pi(\Lambda).x, \quad x \ominus \Lambda = x.\pi(\Lambda) \quad (x \in E, \Lambda \in l^1(S)^{**}).$$

It is easy to see that E is a symmetric normal Banach $l^1(S)^{**}$ - $l^1(E)$ -module. Consider the mapping $D \circ \pi : l^1(S)^{**} \rightarrow E$, in Figure 2. We have,

$$\begin{aligned}
 D \circ \pi(\Lambda\Gamma) &= D(\pi(\Lambda)\pi(\Gamma)) \\
 &= D \circ \pi(\Lambda).\chi \circ \pi(\Gamma) + \chi \circ \pi(\Lambda).D \circ \pi(\Gamma) \\
 &= D \circ \pi(\Lambda).\pi(\chi^{**}(\Gamma)) + \pi(\chi^{**}(\Lambda)).D \circ \pi(\Gamma) \\
 &= D \circ \pi(\Lambda) \ominus \chi^{**}(\Gamma) + \chi^{**}(\Lambda) \ominus D \circ \pi(\Gamma).
 \end{aligned}$$

It follows that $D \circ \pi$ is a module χ^{**} -derivation. Since $l^1(S)^{**}$ is χ^{**} -module Connes amenable, by Definition 2.3, there exists $x \in E$ such that for all $\Lambda \in l^1(S)^{**}$,

$$\begin{aligned}
 D \circ \pi(\Lambda) &= \chi^{**}(\Lambda) \ominus x - x \ominus \chi^{**}(\Lambda) \\
 &= \pi(\chi^{**}(\Lambda)).x - x.\pi(\chi^{**}(\Lambda)) \\
 &= \chi(\pi(\Lambda)).x - x.\chi(\pi(\Lambda)).
 \end{aligned}$$

Therefore $D(\delta_s) = \chi(\delta_s).x - x.\chi(\delta_s)$ for all $\delta_s \in l^1(S)$, and hence by (2.5), D is inner. \square

Theorem 2.6. *Let S be a discrete semigroup, $l^1(S)$ be a non-abelian symmetric dual semigroup algebra and $\chi \in \mathcal{H}_{w^*}(l^1(S))$. If $l^1(S)$ is χ -module Connes amenable, then $l^1(S)$ has a bounded approximate identity for $\chi(l^1(S))$.*

Proof. By Theorem 2.5, we assume that $l^1(S)$ is a symmetric Banach $l^1(S)$ - $l^1(E)$ -module whose space of underlying is $l^1(S)$, and acts on $l^1(S)$ via

$$\delta_s \Delta f := \delta_s f, \quad f \Delta \delta_s := 0, \quad (\delta_s \in l^1(S), f \in l^1(S)).$$

Consider the identity map $I : l^1(S) \rightarrow l^1(S)$. By relations (2.3) and (2.4), it is easy to see that $I \circ \chi$ is a module χ -derivation. By the hypothesis, since $l^1(S)$ is χ -module Connes amenable, there exists $e_{l^1(S)} \in l^1(S)$ such that

$$I \circ \chi(\delta_s) = \chi(\delta_s) \Delta e_{l^1(S)} - e_{l^1(S)} \Delta \chi(\delta_s).$$

Therefore, $\chi(\delta_s) = \chi(\delta_s) \Delta e_{l^1(S)}$ and the element $e_{l^1(S)}$ is a bounded approximate identity for $\chi(l^1(S))$. \square

An analogue of the following theorem for commutative dual Banach algebra is given in [8, Theorem 2.5].

Theorem 2.7. *Let S be a discrete semigroup, $l^1(S)$ be a dual semigroup algebra and $\chi \in \mathcal{H}_{w^*}(l^1(S))$. If $l^1(S)$ is χ -module Connes amenable, then $l^1(S)$ is $\eta \circ \chi$ -module Connes amenable for any $\eta \in \mathcal{H}_{w^*}(l^1(S))$.*

Proof. Let E be a symmetric normal Banach $l^1(S)$ - $l^1(E)$ -module and $D : l^1(S) \rightarrow E$ be a module $\eta \circ \chi$ -derivation that is w^* -continuous. It is sufficient to show that D is inner. For this purpose we equip E with the module actions defined by

$$\delta_s \bullet x = \eta(\delta_s).x, \quad x \bullet \delta_s = x.\eta(\delta_s), \quad (\delta_s \in l^1(S), x \in E).$$

Now, by using relations (2.6) and (2.7), E becomes a symmetric normal Banach $l^1(S)$ - $l^1(E)$ -module. We have

$$\begin{aligned} D(\delta_s \delta_t) &= D(\delta_s).\eta \circ \chi(\delta_t) + \eta \circ \chi(\delta_s).D(\delta_t) \\ &= D(\delta_s) \bullet \chi(\delta_t) + \chi(\delta_s) \bullet D(\delta_t). \end{aligned}$$

Thus, there exists $y \in E$ such that

$$D(\delta_s) = y \bullet \chi(\delta_s) - \chi(\delta_s) \bullet y = y.\eta \circ \chi(\delta_s) - \eta \circ \chi(\delta_s).y, \quad (\delta_s \in l^1(S)).$$

The above argument completes the proof. \square

3. χ -MODULE NORMAL VIRTUAL DIAGONALS

Let \mathcal{U} be a Banach algebra. Let S be a discrete semigroup and $l^1(S)$ be a dual Banach algebra that is hold in compatible module actions of (2.2).

Let \mathcal{I} be the closed ideal of the projective tensor product of $l^1(S)$ with itself, $l^1(S) \widehat{\otimes} l^1(S)$. We put

$$\mathcal{I} = \langle \alpha \cdot (\delta_a \otimes \delta_b) - (\delta_a \otimes \delta_b) \cdot \alpha \rangle, \quad (\alpha \in \mathcal{U}, \delta_a, \delta_b \in l^1(S)).$$

We denote the module projective tensor product of $l^1(S)$ and itself by $l^1(S) \widehat{\otimes}_{\mathcal{U}} l^1(S)$, and we consider $l^1(S) \widehat{\otimes}_{\mathcal{U}} l^1(S) \approx \frac{l^1(S) \widehat{\otimes} l^1(S)}{\mathcal{I}}$.

Suppose that $\mathcal{J} = \langle (\alpha \cdot \delta_a) \cdot \delta_b - \delta_a \cdot (\delta_b \cdot \alpha) \rangle$, is the closed ideal of $l^1(S)$. It is clear that $\overline{\mathcal{J}}^{w^*} = \mathcal{J}$. Then by use of [18], the quotient algebra $\frac{l^1(S)}{\mathcal{J}}$ is dual with predual

$${}^{\perp}\mathcal{J} = \{\Phi \in c_0(S) : \langle \Phi, j \rangle = 0 \text{ for all } j \in \mathcal{J}\}.$$

Also, we have

$$\mathcal{J}^{\perp} = \{\Psi \in l^{\infty}(S) : \langle \Psi, j \rangle = 0 \text{ for all } j \in \mathcal{J}\}.$$

Throughout this section, it is assumed that

$$\mathcal{L}_{\omega^*}^2\left(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C}\right) = \left\{ \mathcal{F} \mid \mathcal{F} : \frac{l^1(S)}{\mathcal{J}} \times \frac{l^1(S)}{\mathcal{J}} \rightarrow \mathbb{C} \right\}, \quad (3.1)$$

that \mathcal{F} is separately w^* -continuous two-linear map, and suppose that

$$\Theta : l^1(S) \widehat{\otimes}_{\mathcal{U}} l^1(S) \rightarrow \frac{l^1(S)}{\mathcal{J}}$$

is the multiplication operator with $\Theta(\delta_s \otimes \delta_t + \mathcal{I}) = \delta_s \delta_t + \mathcal{J}$. Using the Open Mapping Theorem the quotient map Θ^* , maps ${}^{\perp}\mathcal{J}$ onto $\mathcal{L}_{\omega^*}^2\left(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C}\right)$. It follows that Θ^{**} drops to an $l^1(S)$ - \mathcal{U} -module homomorphism

$$\Theta^{**} : \mathcal{L}_{\omega^*}^2\left(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C}\right)^* \rightarrow \frac{l^1(S)}{\mathcal{J}}.$$

Also, consider the map that is defined by

$$\tilde{\chi} : \frac{l^1(S)}{\mathcal{J}} \rightarrow \frac{l^1(S)}{\mathcal{J}}, \quad \tilde{\chi}(\delta_s + \mathcal{J}) = \chi(\delta_s) + \mathcal{J}, \quad \delta_s \in l^1(S).$$

Definition 3.1. Let S be a discrete semigroup, $l^1(S)$ be a dual Banach algebra and $\chi : l^1(S) \rightarrow l^1(S)$ be a bounded w^* -continuous module homomorphism. An element $\mathfrak{D} \in \mathcal{L}_{\omega^*}^2\left(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C}\right)^*$ is called a χ -module normal virtual diagonal for $l^1(S)$ if

$$\Theta^{**}(\mathfrak{D}) \cdot (\chi(\delta_q) + \mathcal{J}) = (\chi(\delta_q) + \mathcal{J}) \quad (\delta_q \in l^1(S)), \quad (3.2)$$

and

$$\mathfrak{D} \cdot (\chi(\delta_q) + \mathcal{J}) = (\chi(\delta_q) + \mathcal{J}) \cdot \mathfrak{D} \quad (\delta_q \in l^1(S)). \quad (3.3)$$

In the sequel, we prove the main theorem of this section. Indeed, we show that if $l^1(S)$ is an unital dual Banach semigroup algebra with an *id*-module normal virtual diagonal, then $l^1(S)$ is module Connes amenable.

Theorem 3.2. *Let S be a discrete semigroup and \mathcal{U} be a dual Banach algebra. Let $l^1(S)$ be an unital dual Banach \mathcal{U} -module that contain an *id*-module normal virtual diagonal. Then $l^1(S)$ is *id*-module Connes amenable.*

Proof. Let X be a symmetric normal Banach $l^1(S)$ - \mathcal{U} -module. First, we note that $l^1(S)$ has an identity. It is sufficient to show that $l^1(S)$ is *id*-module Connes amenable. We suppose that X is unital. Let $D : l^1(S) \rightarrow X$ be a w^* -continuous module derivation. It is clear that X is a normal Banach $\frac{l^1(S)}{\mathcal{J}}$ - \mathcal{U} -module. Let $X = (X_*)^*$. Since X is symmetric, D vanishes on \mathcal{J} . We define

$$\tilde{D} : \frac{l^1(S)}{\mathcal{J}} \rightarrow X, \quad \tilde{D}(\delta_s + \mathcal{J}) := D(\delta_s) \quad (\delta_s \in l^1(S)).$$

For every $x \in X_*$, we correspond $\Omega_x : \frac{l^1(S)}{\mathcal{J}} \times \frac{l^1(S)}{\mathcal{J}} \rightarrow \mathbb{C}$ via

$$\Omega_x(\delta_s + \mathcal{J}, \delta_t + \mathcal{J}) = \langle x, (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J}) \rangle \quad (\delta_s, \delta_t \in l^1(S)). \quad (3.4)$$

It is clear that $\Omega_x \in \mathcal{L}_{w^*}^2(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C})$. By using (3.1) and Definition 3.1, suppose that $\mathcal{F} \in \mathcal{L}_{w^*}^2(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C})$ and $\mathfrak{D} \in \mathcal{L}_{w^*}^2(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C})^*$. We define

$$\langle \mathfrak{D}, \mathcal{F} \rangle = \int_{l^1(S) \widehat{\otimes}_{\mathcal{U}} l^1(S)} \mathcal{F}(\delta_s + \mathcal{J}, \delta_t + \mathcal{J}) d\mathfrak{D}(\delta_s + \mathcal{J}, \delta_t + \mathcal{J}). \quad (3.5)$$

More generally, suppose that X is a dual Banach space. From Definition 3.1 and above notation it follows that for $\mathcal{F} \in \mathcal{L}_{w^*}^2(\frac{l^1(S)}{\mathcal{J}}, \mathbb{C})$,

$$\mathfrak{D} \cdot (\delta_q + \mathcal{J}) = (\delta_q + \mathcal{J}) \cdot \mathfrak{D}$$

is equivalent to

$$\int \mathcal{F}(\delta_q \delta_s + \mathcal{J}, \delta_t + \mathcal{J}) d\mathfrak{D} = \int \mathcal{F}(\delta_s + \mathcal{J}, \delta_t \delta_q + \mathcal{J}) d\mathfrak{D}.$$

Suppose that $\mathcal{G} : \frac{l^1(S)}{\mathcal{J}} \times \frac{l^1(S)}{\mathcal{J}} \rightarrow X$ is a bilinear map such that

$$\delta_s + \mathcal{J} \longrightarrow \mathcal{G}(\delta_s + \mathcal{J}, \delta_t + \mathcal{J}), \quad \delta_t + \mathcal{J} \longrightarrow \mathcal{G}(\delta_s + \mathcal{J}, \delta_t + \mathcal{J})$$

are w^* -continuous. Now for $x \in X_*$ and $\int \mathcal{G}d\mathfrak{D} \in X$ define

$$\left\langle \int \mathcal{G}d\mathfrak{D}, x \right\rangle = \int \left\langle \mathcal{G}(\delta_s + \mathcal{J}, \delta_t + \mathcal{J}), x \right\rangle d\mathfrak{D}(\delta_s + \mathcal{J}, \delta_t + \mathcal{J}).$$

By using (3.2) and (3.3), for each $\delta_s, \delta_t \in l^1(S)$ and $\Phi \in c_0(S)$ we have

$$\left\langle \int \delta_s \delta_t + \mathcal{J}d\mathfrak{D}, \Phi + \mathcal{J}^\perp \right\rangle = \left\langle \mathfrak{D}, \Theta^*(\Phi + \mathcal{J}^\perp) \right\rangle = \left\langle \Theta^{**}(\mathfrak{D}), \Phi + \mathcal{J}^\perp \right\rangle.$$

Now, put

$$H(x) = \langle \mathfrak{D}, \Omega_x \rangle \quad (x \in X_*). \quad (3.6)$$

Let $\delta_q \in l^1(S)$, by relations (3.5) and (3.6), we obtain

$$\begin{aligned} \left\langle (\delta_q + \mathcal{J}).H - H.(\delta_q + \mathcal{J}), x \right\rangle &= \left\langle (H, x.(\delta_q + \mathcal{J}) - x.(\delta_q + \mathcal{J})) \right\rangle \\ &= \int \left\langle (\delta_s \delta_t + \mathcal{J})\tilde{D}(\delta_q + \mathcal{J}), x \right\rangle d\mathfrak{D} \\ &= \left\langle D, \Omega_{x.(\delta_q + \mathcal{J}) - (\delta_q + \mathcal{J}).x} \right\rangle \\ &= \int \Omega_{x.(\delta_q + \mathcal{J}) - (\delta_q + \mathcal{J}).x} d\mathfrak{D} \\ &= \int \left\langle x.(\delta_q + \mathcal{J}) - (\delta_q + \mathcal{J}).x, (\delta_s \right. \\ &\quad \left. + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J}) \right\rangle d\mathfrak{D} \\ &= \int \left\langle x, (\delta_q + \mathcal{J})(\delta_s + \mathcal{J})\tilde{D} \right\rangle (\delta_t + \mathcal{J}) \\ &\quad - (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J}) \right\rangle d\mathfrak{D} \\ &= \int \left\langle x, (\delta_q \delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J}) \right. \\ &\quad \left. - (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J}) \right\rangle d\mathfrak{D} \\ &= \int \left\langle (x, (\delta_s + \mathcal{J})\tilde{D}(\delta_t \delta_q + \mathcal{J}) \right. \\ &\quad \left. - (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J}) \right\rangle d\mathfrak{D} \\ &= \int \left\langle (x, (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J}) \right. \end{aligned}$$

$$\begin{aligned} & - (\delta_s + \mathcal{J})\tilde{D}(\delta_t + \mathcal{J})(\delta_q + \mathcal{J}) \\ & + (\delta_s + \mathcal{J})(\delta_t + \mathcal{J})\tilde{D}(\delta_q + \mathcal{J}) \rangle d\mathfrak{D}. \end{aligned}$$

Thus we have

$$\begin{aligned} \langle (\delta_q + \mathcal{J}).H - H.(\delta_q + \mathcal{J}), x \rangle &= \int \langle (\delta_s + \mathcal{J})(\delta_t + \mathcal{J})\tilde{D}(\delta_q + \mathcal{J}), x \rangle d\mathfrak{D} \\ &= \int \langle (\delta_s\delta_t + \mathcal{J})\tilde{D}(\delta_q + \mathcal{J}), x \rangle d\mathfrak{D} \\ &= \int \langle (\delta_s\delta_t + \mathcal{J}), x \rangle d\mathfrak{D} \cdot \tilde{D}(\delta_q + \mathcal{J}) \\ &= \int \langle \Theta^{**}(\mathfrak{D}).\tilde{D}(\delta_q + \mathcal{J}), x \rangle \end{aligned}$$

Therefore, $D(\delta_q) = \delta_q.H - H.\delta_q$ holds. \square

In Theorem 3.2, it is shown that if an unital semigroup algebra $l^1(S)$ has a module normal virtual diagonal, then $l^1(S)$ is module Connes amenable.

In the next section we prove an important inherited property for above theorem.

4. SOME INHERITED PROPERTIES

Before turning our result, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual semigroup algebra with predual $c_0(S)$, i.e $l^1(S) = (c_0(S))^*$ [4, Theorem 4.6].

Lemma 4.1. *Let S be a weakly cancellative semigroup with idempotents E and $l^1(S)$ be a dual Arens regular semigroup algebra. Moreover, let $l^1(S)$ be a dual Banach $l^1(E)$ -module. If $\chi \in \mathcal{H}_{w^*}(l^1(S))$, then the following conditions are equivalent:*

- (1) $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ is $\chi\widehat{\otimes}_{l^1(E)}\chi$ -module Connes amenable,
- (2) $l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**}$ is $\chi^{**}\widehat{\otimes}_{l^1(E)}\chi^{**}$ -module Connes amenable.

Proof. Let $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ be $\chi\widehat{\otimes}_{l^1(E)}\chi$ -module Connes amenable and E be a symmetric normal Banach $l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**}$ -module. Moreover, let $D : l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**} \rightarrow E$ be a w^* -continuous module $\chi^{**}\widehat{\otimes}_{l^1(E)}\chi^{**}$ -derivation. Let

$$\mu : l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S) \rightarrow l^1(S)^{**}\widehat{\otimes}_{l^1(E)}l^1(S)^{**}$$

be the canonical map. It is clear that μ is w^* -continuous. Without loss of generality, by [5, Theorem 1.3] suppose that S is a finite semigroup. Since $l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S)$ is a dual Banach $l^1(E)$ -module, by the hypothesis and by using Theorem 2.5 the proof of the first part is complete. For the converse, we apply the second part of Theorem 2.5. \square

Remark 4.2. In fact, the proof of Lemma 4.1 say that for any symmetric normal Banach $l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S)$ - $l^1(E)$ -module E , each w^* -continuous module $\chi \widehat{\otimes}_{l^1(E)} \chi$ -derivation from $l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S)$ to E is inner if and only if for any symmetric normal Banach $l^1(S)^{**} \widehat{\otimes}_{l^1(E)} l^1(S)^{**}$ - $l^1(E)$ -module E , each w^* -continuous module $\chi^{**} \widehat{\otimes}_{l^1(E)} \chi^{**}$ -derivation from module projective tensor product $l^1(S)^{**} \widehat{\otimes}_{l^1(E)} l^1(S)^{**}$ to E is inner where, S is a weakly cancellative semigroup with idempotents E .

Corollary 4.3. *Under the conditions of Lemma 4.1, the module projective tensor product $\overbrace{l^1(S) \widehat{\otimes}_{l^1(E)} \cdots \widehat{\otimes}_{l^1(E)} l^1(S)}^n$ is $\chi \widehat{\otimes}_{l^1(E)} \cdots \widehat{\otimes}_{l^1(E)} \chi$ -module Connes amenable if and only if $\overbrace{l^1(S)^{**} \widehat{\otimes}_{l^1(E)} \cdots \widehat{\otimes}_{l^1(E)} l^1(S)^{**}}^n$ is $\chi^{**} \widehat{\otimes}_{l^1(E)} \cdots \widehat{\otimes}_{l^1(E)} \chi^{**}$ -module Connes amenable.*

Theorem 4.4. *Let S be a weakly cancellative semigroup with idempotents E , $l^1(S)$ be an unital dual Banach $l^1(E)$ -module and $l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S)$ be a dual Banach $l^1(E)$ -module. If $\chi \in \mathcal{H}_{w^*}(l^1(S))$ and $l^1(S)$ is χ -module Connes amenable, then $l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S)$ is $\chi \widehat{\otimes}_{l^1(E)} \chi$ -module Connes amenable.*

Proof. Consider the following commutative diagram,

$$\begin{array}{ccc}
 l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S) & & \\
 \uparrow \kappa & \searrow \widehat{D} & \\
 l^1(S) \widehat{\otimes} l^1(S) & \xrightarrow{D = \widehat{D} \circ \pi} & E
 \end{array}$$

FIGURE 3. Commutative diagram.

Let $l^1(S)$ be χ -module Connes amenable. Let E be a symmetric normal Banach $l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S)$ - $l^1(E)$ module and

$$\widehat{D} : l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S) \rightarrow E$$

be a module $\chi \otimes_{l^1(E)} \chi$ -derivation that is w^* -continuous. Consider the quotient map $\pi : l^1(S) \widehat{\otimes} l^1(S) \rightarrow l^1(S) \widehat{\otimes}_{l^1(E)} l^1(S)$. Define

$$\delta_s \otimes \delta_t . x = \pi(\delta_s \otimes \delta_t) \otimes x, \quad x . (\delta_s \otimes \delta_t) = x \otimes \pi(\delta_s \otimes \delta_t)$$

for every $\delta_s, \delta_t \in l^1(S)$ and $x \in E$. Since π is w^* -continuous, E is a normal Banach $l^1(S) \widehat{\otimes} l^1(S)$ - $l^1(E)$ module. Put

$$\widehat{D} \circ \pi : l^1(S) \widehat{\otimes} l^1(S) \rightarrow E.$$

It is clear that $D = \widehat{D} \circ \pi$ is w^* -continuous module $\chi \otimes \chi$ -derivation. It is known that if $\widehat{D} \circ \pi$ is inner, then \widehat{D} is inner. For this purpose, suppose that $e_{l^1(S)}$ is an identity element for $l^1(S)$. We define

$$\delta_s \blacktriangleleft x = (\delta_s \otimes e_{l^1(S)})x, \quad x \blacktriangleleft \delta_s = x(\delta_s \otimes e_{l^1(S)}) \quad (\delta_s \in l^1(S), x \in E).$$

For $\delta_s \in l^1(S), x \in E$ and $\delta_u \in l^1(E)$, we obtain

$$\begin{aligned} \delta_s \blacktriangleleft (\delta_u . x) - (\delta_s . \delta_u) \blacktriangleleft x &= (\delta_s \otimes e_{l^1(S)}) . (\delta_u . x) - (\delta_s . \delta_u \otimes e_{l^1(S)}) . x \\ &= (\delta_s \otimes e_{l^1(S)}) . (\delta_u . x) - (\delta_u . \delta_s \otimes e_{l^1(S)}) . x \\ &= (\delta_s \otimes e_{l^1(S)}) . (\delta_u . x) - (\delta_u . (\delta_s \otimes e_{l^1(S)})) . x \\ &= (\delta_s \otimes e_{l^1(S)}) . (\delta_u . x) - ((\delta_s \otimes e_{l^1(S)}) . \delta_u) . x \\ &= (\delta_s \otimes e_{l^1(S)}) . (\delta_u . x) - ((\delta_s \otimes e_{l^1(S)}) . (\delta_u . x)) \\ &= 0. \end{aligned}$$

Also, the same argument is hold for the right actions. Therefore, E is a symmetric normal Banach $l^1(S)$ - $l^1(E)$ -bimodule. Put

$$D_{l^1(S)} : l^1(S) \rightarrow E, \quad D_{l^1(S)}(\delta_s) = D(\delta_s \otimes e_{l^1(S)}) \quad (\delta_s \in l^1(S)).$$

We obtain

$$\begin{aligned} D_{l^1(S)}(\delta_s \delta_t) &= D(\delta_s \otimes e_{l^1(S)}) . \chi \otimes \chi(\delta_t \otimes e_{l^1(S)}) \\ &\quad + \chi \otimes \chi(\delta_s \otimes e_{l^1(S)}) . D(\delta_t \otimes e_{l^1(S)}) \\ &= D_{l^1(S)}(\delta_s) \blacktriangleleft \chi(\delta_t) + \chi(\delta_s) \blacktriangleleft D_{l^1(S)}(\delta_t). \end{aligned}$$

Since $l^1(S)$ is χ -module Connes amenable, there is $g \in E$ such that $D_{l^1(S)} = ad_g$. Thus, $\tilde{D} = D - ad_g$ vanishes on $l^1(S) \otimes e_{l^1(S)}$. Setting

$$\delta_s \blacktriangleright x = (e_{l^1(S)} \otimes \delta_s) . x, \quad x \blacktriangleright \delta_s = x . (e_{l^1(S)} \otimes \delta_s) \quad (4.1)$$

for every $\delta_s \in l^1(S)$ and $x \in E$. The module actions defined in (4.1), makes E into an $l^1(S)$ - $l^1(E)$ -bimodule for $\delta_s \in l^1(S)$ and $x \in E$. Let us now,

$D'_{l^1(S)}(\delta_s) := \tilde{D}(e_{l^1(S)} \otimes \delta_s)$. Set

$$\mathcal{Y} = \{y \in E_* : \langle \tilde{D}(e_{l^1(S)} \otimes \delta_s), y \rangle = 0\}.$$

Since \tilde{D} is w^* -continuous, by use of [21, Theorem 4.9] and applying the similar argument we conclude $(\frac{E_*}{\mathcal{Y}})^* = \overline{\tilde{D}(e_{l^1(S)} \otimes \delta_s)}^{w^*}$. Furthermore, $\overline{\tilde{D}(e_{l^1(S)} \otimes \delta_s)}^{w^*}$ is a w^* -closed submodule of E . Thus, $\overline{\tilde{D}(e_{l^1(S)} \otimes \delta_s)}^{w^*}$ is a symmetric normal Banach $l^1(S)$ - $l^1(E)$ -module. By hypothesis, there exists $\delta_f \in \overline{\tilde{D}(e_{l^1(S)} \otimes \delta_s)}^{w^*}$ such that

$$\begin{aligned} \tilde{D}(e_{l^1(S)} \otimes \delta_s) &= D'_{l^1(S)}(\delta_s) = \chi(\delta_s) \blacktriangleright \delta_f - \delta_f \blacktriangleright \chi(\delta_s) \\ &= \chi \otimes_{l^1(S)} \chi(e_{l^1(S)} \otimes \delta_s) \cdot \delta_f - \delta_f \cdot \chi \otimes_{l^1(S)} \chi(e_{l^1(S)} \otimes \delta_s) \end{aligned}$$

and $\tilde{D} - ad_{\delta_f} |_{(e_{l^1(S)} \otimes l^1(S))}$ is vanishes on $e_{l^1(S)} \otimes l^1(S)$. All in all,

$$\tilde{D} - ad_{\delta_f} = D - ad_{\delta_g} - ad_{\delta_f} |_{l^1(S) \hat{\otimes}_{l^1(E)} l^1(S)} = 0.$$

□

Example 4.5. Suppose that \mathbb{N} is the set of positive integers,

- (1) It is well-known that the semigroup algebra $l^1(\mathbb{N}, \max)$ has an identity. Since \mathbb{N} is weakly cancellative, $l^1(\mathbb{N}, \max)$ is a dual semigroup algebra such that $l^1(\mathbb{N}, \max) = (c_0(\mathbb{N}))^*$. By [6, Theorem 5.13], $l^1(\mathbb{N}, \max)$ is not Connes amenable. Moreover, $l^1(\mathbb{N}, \max)$ is module amenable on $l^1(E_{(\mathbb{N}, \max)})$, so it is *id*-module Connes amenable. Using Theorem 2.6, it has a bounded approximate identity for $\chi(l^1(\mathbb{N}, \max))$.
- (2) If $S = (\mathbb{N}, \min)$, then the semigroup algebra $l^1(\mathbb{N}, \min)$ is a non-unital Banach algebra with a bounded approximate identity. Furthermore, $l^1(\mathbb{N}, \min)$ is module amenable on $l^1(E_{(\mathbb{N}, \min)})$, so it is *id*-module Connes amenable.

Acknowledgments

We thank the referees for their time and valuable comments.

REFERENCES

1. M. Amini, Module amenability for semigroup algebras, *Semigroup Forum*, **69** (2004), 243–254.
2. M. Amini, A. Bodaghi and D. E. Bagha, Module amenability of the second dual and module topological center of semigroup algebras, *Semigroup Forum*, **80** (2010), 302–312.

3. M. Amini and R. Rezavand, Module operator virtual diagonals on the Fourier algebra of an inverse semigroup, *Semigroup Forum*, **97** (2018), 562–570.
4. H. G. Dales, A. T.-M. Lau and D. Strauss, *Banach algebras on Semigroups and on their compactifications*, American Mathematical Society, 2010.
5. H. G. Dales and D. Strauss, Arens regularity for totally ordered semigroups, *Semigroup Forum*, **105**(1) (2022), 172–190.
6. M. Daws, Connes amenability of bidual and weighted semigroup algebras, *Math. Scand.*, **99** (2006), 217–246.
7. A. Ghaffari and S. Javadi, φ -Connes amenability of dual Banach algebras, *Bull. Iranian Math. Soc.*, **43** (2017), 25–39.
8. A. Ghaffari, S. Javadi and E. Tamimi, φ - module Connes amenability of dual Banach Algebras, *J. Algebr. Syst.*, **8**(1) (2020), 69–82.
9. B. E. Johnson, Cohomology in semigroup Algebras, *Mem. Amer. Math. Soc.*, **127** (1972).
10. A. T.-M. Lau, Normal Structure and Common Fixed Point Properties for Semigroups of Nonexpansive Mappings in Banach Spaces, *Fixed Point Theory and Applications*, (2010), 1–14.
11. A. T.-M. Lau and Y. Zhang, Algebraic and analytic properties of semigroups related to fixed point properties of non-expansive mappings, *Semigroup Forum*, (2020), 1–26.
12. A. T.-M. Lau and Y. Zhang, Fixed point properties for semigroups of nonexpansive mappings on convex sets in dual Banach spaces, *Ann. Univ. Paedagog. Crac. Stud. Math.*, **17** (2018), 67–87.
13. A. T.-M. Lau and Y. Zhang, Fixed point properties of semigroups of non-expansive mappings, *J. Funct. Anal.*, **254** (2008), 2534–2554.
14. A. Minapoor, Ideal Connes amenability of l^1 -Munn algebras and its application to semigroup algebras, *Semigroup Forum*, **102**(4) (2021), 756–764.
15. A. Minapoor, A. Bodagh and O. T. Mewomo, Ideal Connes-amenability of Lau product of Banach algebras, *Eurasian Math. J.*, **12**(4) (2021), 74–81.
16. W. D. Munn, A class of irreducible matrix representations of an arbitrary inverse semigroup, *Proc. Glasgow Math. Assoc.*, **5** (1961), 41–48.
17. R. Rezavand, M. Amini, M. H. Sattari and D. E. Bagha, Module Arens regularity for semigroup algebras, *Semigroup Forum*, **77** (2008), 300–305.
18. W. Rudin, *Functional Analysis*, McGraw Hill, New York, 1991.
19. V. Runde, Amenability for dual Banach algebras, *Studia Math.*, **148** (2001), 47–66.
20. V. Runde, Connes amenability and normal virtual diagonals for measure algebras I, *J. London Math. Soc.*, **67** (2003), 643–656.
21. V. Runde, *Lectures on Amenability*, Springer New York, 2002.

Ebrahim Tamimi

Department of Mathematics, Velayat University, P.O. Box 9917638733, Iranshahr, Iran.
Email: e.tamimi@velayat.ac.ir

Ali Ghaffari

Department of Mathematics, University of Semnan, P.O. Box 35195-363, Semnan, Iran.
Email: aghaffari@semnan.ac.ir

A SURVEY ON χ -MODULE CONNES AMENABILITY
OF SEMIGROUP ALGEBRAS

E. TAMIMI AND A. GHAFFARI

بررسی کان میانگین پذیری χ -مدولی جبرهای نیم‌گروهی

ابراهیم تمیمی^۱ و علی غفاری^۲

^۱گروه ریاضی محض، دانشکده علوم ریاضی، دانشگاه ولایت، ایرانشهر، ایران

^۲گروه ریاضی محض، دانشکده علوم ریاضی، دانشگاه سمنان، سمنان، ایران

در این مقاله، کان میانگین‌پذیری χ -مدولی جبر نیم‌گروهی $l^1(S)$ را مطالعه می‌کنیم که در آن χ یک هم‌ریختی مدولی کراندار از $l^1(S)$ به $l^1(S)$ بوده که ضعیف ستاره پیوسته است. فرض کنید S یک نیم‌گروه حذفی ضعیف وارون با زیرنیم‌گروه E از خودتوان‌ها باشد. مطالعه‌ی قطرهای واقعی نرمال χ -مدولی از اهداف این مقاله است. کان میانگین‌پذیری χ -مدولی جبر نیم‌گروهی $l^1(S)$ را مشخص‌سازی می‌کنیم. همچنین، اگر $l^1(S)$ به عنوان یک مدول باناخ روی $l^1(E)$ دارای یک قطر واقعی نرمال id -مدولی باشد آن‌گاه کان میانگین‌پذیر id -مدولی خواهد بود. مشخص‌سازی‌های دیگری از کان میانگین‌پذیری χ -مدولی $l^1(S)$ ارائه شده است.

کلمات کلیدی: کان میانگین‌پذیر χ -مدولی، اشتقاق χ -مدولی، قطر واقعی نرمال χ -مدولی، نیم‌گروه حذفی ضعیف.