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# On some properties of self-similar action of a group on a $k$-graph

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**Abstract**

In this paper, we study the basic properties of the notion of a self-similar action of a group $G$ on a $k$-graph. Also, by considering a self-similar $k$-graph over a certain group, we prove some hereditary properties through of so called restriction mapping on $k$-graph.

**Keywords:** Self-similar action, $k$-graph, $G$-hereditary, $G$-strongly connected, restriction map

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## 1 Introduction

Let $G$ be a discrete (countable) group, and $(E, r, s)$ be a finite directed graph with no sources. In [3], Exel-Pardo introduced a notion of a selfsimilar action of $G$ on $E$, which naturally generalizes the notion of selfsimilar groups (see, e.g., [5]). For investigate and to study a certain $C^*$-algebra, Exel-Pardo in [3] associated the self-similar action $(G, E)$ an inverse semigroup $S_{G,E}$.

In this paper, we study the properties of the notion of a self-similar action of a group $G$ on a $k$-graph. We know that for higher-dimensional cases, in general the construction of the inverse semigroup $S_{G,E}$ in [3] mentioned above does not apply, and so unlike [3] one can not apply the machinery in [1, 2].

## 2 Main results

**Definition 2.1.** Let $\Lambda$ be a $k$-graph, and $G$ be a group acting on $\Lambda$. Then the action is said to be self-similar if there exists a restriction map $G \times \Lambda \to G$, $(g, \mu) \mapsto g\,|_\mu$, such that

(i) $g.(\mu\nu) = (g.\mu)(g\,|_\mu .\nu)$ for all $g \in G, \mu, \nu \in \Lambda$ with $s(\mu) = r(\nu)$.

(ii) $g\,|_\nu = g$ for all $g \in G$, $\nu \in \Lambda^0$;

(iii) $g\,|_{\mu\nu} = g\,|_\mu\,|_\nu$ for all $g \in G, \mu, \nu \in \Lambda$ with $s(\mu) = r(\nu)$;

(iv) $1_G\,|_\mu = 1_G$ for all $\mu \in \Lambda$;

(v) $(gh)\,|_\mu = g\,|_{h.\mu}\, h\,|_\mu$ for all $g, h \in G$, $\mu \in \Lambda$.

Moreover, $\Lambda$ and $G$ are called a self-similar $k$-graph over $G$ and self-similar group on $\Lambda$, respectively [4].

[1] speaker

**Definition 2.2.** Let $\Lambda$ be a self-similar $k$-graph over a group $G$, and $H$ be a subset of $\Lambda^0$. Then $H$ is said to be $G$-hereditary if $s(H\Lambda) \subseteq H$ and $G.H \subseteq H$. Notice that $H$ is hereditary in the usual sense, and that it naturally induces a self-similar action of $G$ on $H\Lambda$ [4].

**Definition 2.3.** Let $\Lambda$ be a self-similar $k$-graph over a group $G$. Then $\Lambda$ is said to be $G$-strongly connected if $(G.\nu)\Lambda\omega \neq \varnothing$ for all $\nu, \omega \in \Lambda^0$.

Clearly, if $\Lambda$ is strongly connected, then it is $G$-strongly connected [4].

**Theorem 2.4.** *Let $\Lambda$ be a self-similar $k$-graph over a group $G$. Then*

*(i) $(g+g') \mid_\mu .s(\nu) = (g+g').s(\nu)$ for all $g, g' \in G$, $\mu, \nu \in \Lambda$ with $s(\mu) = r(\nu)$;*

*(ii) $(g+g') \mid_\mu .s(\mu) = (g+g').s(\mu)$ for all $\mu \in \Lambda$;*

*(iii) $((g+g') \mid_\mu)^{-1} = (g+g')^{-1} \mid_{(g+g').\mu}$ for all $g, g' \in G, \mu \in \Lambda$.*

*Proof.* (i) By hypothesis we have

$$\begin{aligned}(g+g').s(\nu) &= (g+g').s(\mu\nu)\\ &= s((g+g').(\mu\nu))\\ &= s((g+g') \mid_\mu.\nu)\\ &= (g+g') \mid_\mu.s(\nu).\end{aligned}$$

(ii) This is a special case of (i).

(iii) We obtain

$$\begin{aligned}((g+g') \mid_\mu)((g+g')^{-1} \mid_{(g+g').\mu}) &= ((g+g')(g+g')^{-1}) \mid_{(g+g').\mu}\\ &= 1_G = ((g+g')^{-1}(g+g')) \mid_\mu\\ &= ((g+g')^{-1} \mid_{(g+g').\mu})((g+g') \mid_\mu).\end{aligned}$$

□

Let $\Lambda$ be a $k$-graph. Set $\Lambda^e := \Lambda^{e_1} \cup \Lambda^{e_2} \cup \cdots \cup \Lambda^{e_k}$. By the next theorem we extend an action of $G$ on $\Lambda^0 \bigcup \Lambda^e$ with a restriction to a self-similar action of $G$ on $\Lambda$.

**Theorem 2.5.** *Let $\Lambda$ be a $k$-graph, and $G$ be a group. Suppose that $G$ acts on the set $\Lambda^0 \bigcup \Lambda^e$, and that there is a restriction map $G \times (\Lambda^0 \bigcup \Lambda^e) \to G$, $(g, x) \mapsto g \mid_x$, satisfying the following properties:*

*(i) $G.\Lambda^n \subseteq \Lambda^n$ for all $n \in \{0, e_i : 1 \leq i \leq k\}$;*

*(ii) $s((g+g').\mu) = (g+g').s(\mu)$ and $r((g+g').\mu) = (g+g').r(\mu)$ for all $g, g' \in G, \mu \in \Lambda^e$;*

*(iii) $(g+g') \mid_\mu .s(\nu) = (g+g').s(\nu)$ for all $g, g' \in G, \mu \in \Lambda^e, \nu \in \Lambda$ with $s(\mu) = r(\nu)$;*

*(iv) $((g+g').\mu)((g+g') \mid_\mu .\nu) = ((g+g').\alpha)((g+g') \mid_\alpha .\beta)$ for all $g, g' \in G, \mu, \nu, \alpha, \beta \in \Lambda^e$ with $\mu = \alpha\beta$;*

*(v) $(g+g') \mid_\nu = (g+g')$ for all $g, g' \in G, \nu \in \Lambda^0$;*

*(vi) $(g+g')|_\mu \mid_\nu = (g+g')|_\alpha|_\beta$ for all $g, g' \in G$, $\mu, \nu, \alpha, \beta \in \Lambda^e$ with $\mu\nu = \alpha\beta$;*

*(vii) $((g+g')h) \mid_\mu = ((g+g') \mid_{h.\mu})(h \mid_\mu)$ for all $g, g', h \in G, \mu \in \Lambda^e$.*

*Then there exists a unique self-similar action of $G$ on $\Lambda$ with the restriction map $\mid$ from $G \times \Lambda$ into $G$ extending the given action and the given map $\mid$.*

*Proof.* Let $\mu \in \Lambda$ with $|\mu| = 2$. We set $\mu = \mu_1\mu_2$ with $\mu_1, \mu_2 \in \Lambda^e$. For $g, g' \in G$, set

$$(g + g').\mu := ((g + g').\mu_1)((g + g') |_{\mu_1} .\mu_2)$$

and

$$(g + g') |_{\mu}:= (g + g') |_{\mu_1} |_{\mu_2}.$$

We can see that both $(g + g').\mu$ and $(g + g') |_{\mu}$ are well-defined. Inductively, we extend the given action and restriction to $\Lambda$. One can see that they satisfy Definition 2.1 $(i) - (v)$ are satisfied. Thus the extensions follows a self-similar action of $G$ on $\Lambda$. □

**Remark 2.6.** Let $\Lambda$ be a self-similar $k$-graph over a group $G$. For $g, g' \in G$ and $x \in \Lambda^\infty$, we define

$$((g + g').x)(p, q) := (g + g') |_{x(0,p)} .x(p, q) \qquad forall \qquad (p, q) \in \Omega_k,$$

and

$$(g + g') |_x (p) := (g + g') |_{x(0,p)} \qquad forall \qquad p \in \mathbb{N}^k.$$

Then $(g + g').x \in \Lambda^\infty$ and $(g + g') |_x$ is a function from $\mathbb{N}^k$ to $G$.

By the following proposition we give some basic properties of $(g + g').x$ and $(g + g') |_x$.

**Proposition 2.7.** *Let $\Lambda$ be a self-similar $k$-graph over a group $G$. The following statements are hold*

*(i) for $g, g' \in G$, the map $\Lambda^\infty \to \Lambda^\infty$, $x \to (g + g').x$ is a homeomorphism;*

*(ii) $\sigma^p((g + g').x) = (g + g') |_{x(0,p)} .\sigma^p(x)$ for all $g, g' \in G$, $p \in \mathbb{N}^k$ and $x \in \Lambda^\infty$;*

*(iii) $(g + g').(\mu x) = ((g + g').\mu)((g + g') |_{\mu} .x)$ for all $g, g' \in G$, $\mu \in \Lambda$ and $x \in s(\mu)\Lambda^\infty$.*

*Proof.* The proof of (i) is straightforward.

(ii) Suppose that $p \in \mathbb{N}^k$ and $(s, t) \in k$, repeatedly using Definition 2.1 (5) follows that

$$\begin{aligned}\sigma^p((g + g').x)(s, t) &= ((g + g').x)(s + p, t + p)\\ &= (g + g') |_{x(0,s+p)} .x(s + p, t + p)\\ &= (g + g') |_{x(0,s+p)} .\sigma^p(x)(s, t)\\ &= (g + g') |_{x(0,p)} |\sigma^p(x)(0, s).\sigma^p(x)(s, t)\\ &= ((g + g') |_{x(0,p)} .\sigma^p(x))(s, t).\end{aligned}$$

The proof of (iii) is straightforward. □

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