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φ-SPLITTING AND φ-CONNES MODULE AMENABILITY IN BANACH ALGEBS

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ABSTRACT. In present paper, we characterize φ-Connes module amenability of a Banach algebra in terms of φ-splitting of the short exact sequence, where φ is continuous bounded module homomorphism whit respect to ω^*-topology on mentioned Banach algebra. Also, we generalize this to module product of two Banach algebras.

1. INTRODUCTION

In [1], Ghaffari et al. studied φ-Connes module amenability of dual Banach algebras The notion of χ-module Connes amenability of semigroup algebras is studied by the authors in [2]. Also, the notion of $\chi \otimes \eta$-strong Connes amenability of certain dual Banach algebras is investigated by Tamimi and Ghaffari in [3]. We know that for Banach algebra $\mathcal{A}$, the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule. A dual Banach $\mathcal{A}$-bimodule E is called normal if the module actions of $\mathcal{A}$ on E are ω^*-continuous. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. Then $\sigma wc(E)$, a closed submodule of E, stands for the set of all elements $x \in E$ such that the following maps are ω^*-ω continuous

$$\mathcal{A} \longrightarrow E; \quad a \longmapsto a.x, \;\; a \longmapsto x.a.$$

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Throughout the paper, $\Delta(\mathcal{A})$ and $\Delta_{\omega^*}(\mathcal{A})$ will denote the sets of all homomorphisms and ω^*-continuous homomorphisms from the Banach algebra $\mathcal{A}$ onto $\mathbb{C}$, respectively.

In present paper, we give a characterization of φ-Connes module amenability of a dual Banach algebra by φ-splitting of the related short exact sequences. Also, by letting that two Banach algebras are φ-Connes module amenable and ψ-Connes module amenable respectively, we show that this property is holds for the special tensor product of their.

2. Preliminary Definitions

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra, and let $3 \leq n \in \mathbb{N}$. A sequence $\mathcal{A}_1 \overset{\varphi_1}{\to} \mathcal{A}_2 \overset{\varphi_2}{\to} .... \overset{\varphi_{n-1}}{\longrightarrow} \mathcal{A}_n$ of $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i \in \{2, ..., n-1\}$ is called exact at position $i = 2, ..., n-1$ if $\varphi_{i-1} = ker\varphi_i$. It is called exact if it is exact at every position $i = 2, ..., n-1$.

We restrict ourselves to exact sequences with few bimodules.

Definition 2.2. Let $\mathcal{A}$ be a Banach algebra. A short exact sequence

$$\Theta : 0 \to \mathcal{A}_1 \overset{\varphi_1}{\to} \mathcal{A}_2 \overset{\varphi_2}{\to} ... \overset{\varphi_{n-1}}{\longrightarrow} \mathcal{A}_n \to 0$$

of Banach $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i = 1, 2, ...n-1$ is admissible, if there exists a bounded linear map $\rho_i : \mathcal{A}_{i+1} \to \mathcal{A}_i$ such that $\rho_i o \varphi_i$ on $\mathcal{A}_i$ for $i = 1, 2, ...n-1$ is the identity map. Further, Θ splits if we may choose ρ_i to be an $\mathcal{A}$-bimodule homomorphism.

Definition 2.3. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be an unital dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$. We say that $\sum \varphi$-splits if there exists a bounded linear map $\rho : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \to \mathcal{A}_*$ such that $\rho o \pi^*(\varphi) = \varphi$ and $\rho(T.a) = \varphi(a)\rho(T)$, for all $a \in \mathcal{A}$ and $T \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$.

Definition 2.4. Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ is a φ-σwc virtual diagonal for $\mathcal{A}$ if $a.M = \varphi(a)M$ for all $a \in \mathcal{A}$ and $\langle \varphi \otimes \varphi, M \rangle = 1$.

Now, we define the map $\mathcal{A}$-bimodule homomorphism $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ by $\pi(a \otimes b) = ab$. We consider the following short exact sequences, which have three non-zero terms:

$$\sum_{\varphi} : 0 \to \mathcal{A}_* \overset{\pi^*_{\mathcal{A}}}{\longrightarrow} \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*/\pi^*_{\mathcal{A}}(\mathcal{A}_*) \to 0,$$

$$\sum_{\psi} : 0 \to \mathcal{B}_* \xrightarrow{\pi_{\mathcal{B}}^*} \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^* \to \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^*/\pi_{\mathcal{B}}^*(\mathcal{B}_*) \to 0.$$

Definition 2.5. Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. $\mathcal{A}$ is φ-Connes amenable if for every normal φ-bimodule E, every bounded ω^*-continuous derivation $D : \mathcal{A} \to E$ is inner.

3. Characterization of φ-Connes module amenability by φ-splitting

In this section, we investigate the relation between notions of φ-Connes module amenability and φ-splitting.

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let E be a dual Banach $\mathcal{A}$-bimodule. E is called normal if for each $x \in E$, the maps

$$\mathcal{A} \to E; \qquad a \to a.x, \quad a \to x.a$$

are ω^*- continuous. If moreover E is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in E$

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

then E is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly for the right and two sided actions. Also, E is called commutative, if

$$\alpha.x = x.\alpha \qquad (\alpha \in \mathcal{U}, x \in E).$$

A module homomorphism from $\mathcal{A}_*$ to $\mathcal{A}_*$ is a map $\varphi : \mathcal{A}_* \to \mathcal{A}_*$ with

$$\varphi(\alpha.a+b.\beta) = \alpha.\varphi(a)+\varphi(b).\beta, \quad \varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}_*, \alpha, \beta \in \mathcal{U}).$$

Throughout the paper, $\mathcal{HOM}_{\omega^*}^b(\mathcal{A}_*)$ will denotes the space of all bounded module homomorphisms from $\mathcal{A}_*$ to $\mathcal{A}_*$ that are ω^*-continuous.

Definition 3.1. ([1], P. 71) Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\varphi \in \mathcal{HOM}_{\omega^*}^b(\mathcal{A}_*)$. let E be a dual Banach $\mathcal{A}_*$-bimodule. A bounded map $D_{\mathcal{U}} : \mathcal{A}_* \to E$ is called a module φ-derivation if

$$D_{\mathcal{U}}(\alpha.a \pm b.\beta) = \alpha.D_{\mathcal{U}}(a) \pm D_{\mathcal{U}}(b).\beta, \quad D_{\mathcal{U}}(ab) = D_{\mathcal{U}}(a).\varphi(b)+\varphi(a).D_{\mathcal{U}}(b),$$

for every $a, b \in \mathcal{A}_*$ and $\alpha, \beta \in \mathcal{U}$.

Definition 3.2. ([1], P. 71) Let $\mathcal{A}_*$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}_*$ is a Banach $\mathcal{U}$-module and $\varphi \in \mathcal{HOM}_{\omega^*}^b(\mathcal{A}_*)$. $\mathcal{A}_*$ is called φ-Connes module amenable if for any commutative normal Banach $\mathcal{A}_*$-$\mathcal{U}$-module E, each ω^*-continuous module φ-derivation $D_{\mathcal{U}} : \mathcal{A}_* \to E$ is inner.

Theorem 3.3. *Let $\mathcal{A}_*$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A}_*)$. Then $\mathcal{A}_*$ is φ-Connes module amenable if and only if Σ_φ φ-splits.*

Suppose that $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras such that $\mathcal{A}$ and $\mathcal{B}$ be dual Banach $\mathcal{U}$-modules. Let I be the closed ideal of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ generated by elements of the form $\alpha.(a\otimes b)-(a\otimes b).\alpha$ for $a \in \mathcal{A}, b \in \mathcal{B}$ and $\alpha \in \mathcal{U}$. $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ is defined to be the quitiont Banach space $\frac{\mathcal{A}\widehat{\otimes}\mathcal{B}}{I}$.

Theorem 3.4. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}, \mathcal{B}$ be unital dual Banach $\mathcal{U}$-modules and let $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. If $\mathcal{A}, \mathcal{B}$ are φ-Connes module amenable, ψ-Connes module amenable respectively, then $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ is $\varphi\widehat{\otimes}_{\mathcal{U}}\psi$-Connes module amenable.*

Theorem 3.5. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}, \mathcal{B}$ be unital dual Banach $\mathcal{U}$- modules and let $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ is $\varphi\widehat{\otimes}_{\mathcal{U}}\psi$-Connes module amenable if and only if $\Sigma_{\varphi\widehat{\otimes}_{\mathcal{U}}\psi}$ $\varphi\widehat{\otimes}_{\mathcal{U}}\psi$-splits.*

4. Conclusion

In this paper, we studied the relation between φ-splitting and φ-Connes module amenability, where φ is a continuous bounded module homomorphism with respect to ω^*-topology.

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