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ON GENERALIZATIONS OF SYMMETRIC BI-DERIVATIONS ON GROUP ALGEBRAS

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Abstract. In this paper, we investigate generalizations of symmetric bi-derivations on $L_0^\infty(G)^*$. For $k \in \mathbb{N}$, we prove that if $B : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ is a symmetric bi-derivation such that $[B(m,m), m^k] \in Z(L_0^\infty(G)^*)$ for all $m \in L_0^\infty(G)^*$, then $B = 0$. Also, we characterize symmetric generalized biderivations on group algebras.

1. Introduction

Let G denote a locally compact abelian group with a fixed left Haar measure λ. We know that $L^1(G)$ and $L^\infty(G)$ are Banach algebras. Consider that $L^\infty(G)$ is the continuous dual of $L^1(G)$. We denote by $L_0^\infty(G)$ the subspace of $L^\infty(G)$ consisting of all functions $g \in L^\infty(G)$ that vanish at infinity. For every $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$ we define the functional $ng \in L_0^\infty(G)^*$ by

$$\langle ng, \varphi \rangle = \langle n, g\varphi \rangle$$

in which $\langle \varphi, \psi \rangle = \langle g, \varphi * \psi \rangle$ and

$$\varphi * \psi(x) = \int_G \phi(y)\psi(y^{-1}x)d\lambda(y) \tag{1.1}$$

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for all $\varphi, \psi \in L_1(G)$ and $x \in G$. We equippe $L_0^\infty(G)^*$ to the first Arens product "$\cdot$" defined by the formula $\langle m.n, g\rangle = \langle m, ng\rangle$ for all $m, n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)^*$. Then $L_0^\infty(G)^*$ is a Banach algebra with the mentioned product. For more information of $L_0^\infty(G)^*$ see [4]. The notion of symmetric bi-derivations is investigated in [1, 8]. Let $\mathcal{A}$ be an algebra and $B(.,.) : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a symmetric bi-linear mapping; that is, $B(x, y) = B(y, x)$, $B(\alpha x, y) = \alpha B(x, y)$ and $B(x + y, z) = B(x, z) + B(y, z)$ for all $x, y, z \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. The mapping $f : \mathcal{A} \to \mathcal{A}$ defined by $f(x) = B(x, x)$ is called the trace of B. We say that B is called a symmetric bi-derivation if we have

$$B(xy, z) = B(x, z)y + xB(y, z) \tag{1.2}$$

for all $x, y, z \in \mathcal{A}$. Also, B is called a symmetric generalized bi-derivation if there exists a symmetric bi-derivation $\tilde{\mathrm{B}}$ of $\mathcal{A}$ such that

$$B(xy, z) = xB(y, z) + \overline{B}(x, z)y \tag{1.3}$$

for all $x, y, z \in \mathcal{A}$. A symmetric generalized bi-derivation B associated with a symmetric bi-derivation $\tilde{\mathrm{B}}$ is denoted by $B_{\overline{B}}$. For $\kappa \in \mathbb{N}$, a linear mapping $T : \mathcal{A} \to \mathcal{A}$ is called κ-(skew) centralizing if

$$[T(x), x^\kappa] \in Z(\mathcal{A}) \qquad (T(x) \circ x^\kappa \in Z(\mathcal{A})) \tag{1.4}$$

for all $x \in \mathcal{A}$, in a special case, if for every $x \in \mathcal{A}$

$$[T(x), x^\kappa] = 0 \qquad (T(x) \circ x^\kappa = 0));$$

then T is called κ-(skew) commuting, where $Z(\mathcal{A})$ is the center of $\mathcal{A}$, $[x, y] = xy - yx$ and $x \circ y := x.y + y.x$ for all $x, y \in \mathcal{A}$. If, $\kappa = 1$, T is called (skew) centralizing and (skew) commuting, respectively. Symmetric bi-derivations on rings have been introduced and studied by Maksa [5]. Vukman [9] proved that if $B : R \times R \to R$ is a symmetric bi-derivation such that for every $x \in R$

$$[[f(x), x], x] \in Z(R);$$

then $B = 0$, where R is a noncommutative prime ring of characteristic not two and three. He conjectured that if there exists $\kappa \in \mathbb{N}$ such that for every $x \in R$ we have $f_\kappa(x) \in Z(R)$; then $B = 0$, where

$$f_{i+1}(x) = [f_i(x), x]$$

for $i > 1$ and $f_1(x) = f(x)$. In [3], Deng gave an affrmative answer to the Vukman's conjecture. For related results on symmetric bi-derivations on Banach algebras see [7]; see also [2] for study of generalized bi-derivations.

The mapping $B(.,.) : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ defined by $B(m, n) = r.m.n$ is a nonzero bi-derivation. These facts lead us to investigate symmetric bi-derivations on $L_0^\infty(G)^*$. In this paper, we first study symmetric bi-derivations on $L_0^\infty(G)^*$ and prove that they map $L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ into the radical of $L_0^\infty(G)^*$. We also show that if $B : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ is a symmetric bi-derivation and f is κ-centralizing for some $\kappa \in \mathbb{N}$, then B is zero map. In the case that, B is a symmetric generalized bi-derivation, we prove that there exists $\theta \in L_0^\infty(G)^*$ such that $B(m,n) = m.n.\theta$ for all $m.n \in L_0^\infty(G)^*$.

2. Main results

In the sequel, we use the symbols D, for symmetric bi-derivations. The following result is an analogue of [1, Proposition 2.1.] for bi-derivations.

Lemma 2.1. *Let $D_1, D_2 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ be symmetric bi-derivations. Then $D_1 + D_2$ maps $L_0^\infty(G)^* \times L_0^\infty(G)^*$ into the radical of $L_0^\infty(G)^*$.*

Proof. For every $m \in L_0^\infty(G)^*$ we define the mapping $\Delta_m : L_0^\infty(G)^* \to L_0^\infty(G)^*$ by

$$\Delta_m(n) = D(m,n). \tag{2.1}$$

For every $m \in L_0^\infty(G)^*$, Δ_m is a derivation on $L_0^\infty(G)^*$ and hence Δ_m maps $L_0^\infty(G)^*$ into its radical for all $m \in L_0^\infty(G)^*$; see [6]. Since $D_i(L_0^\infty(G)^* \times L_0^\infty(G)^*) = \bigcup_m \Delta_m(L_0^\infty(G)^*)$, $i = 1,2$, then D_i maps $L_0^\infty(G)^* \times L_0^\infty(G)^*$ into the radical of $L_0^\infty(G)^*$. □

Theorem 2.2. *Let $D_1, D_2 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ be symmetric bi-derivations, and let f and g be the trace of D_1 and D_2, respectively. Then the following are equivalent.*

(a) *there exists $\kappa \in \mathbb{N}$ such that $f(t^\kappa) = g(t^\kappa) = 0$ for all $t \in L_0^\infty(G)^*$;*
(b) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-commuting;*
(c) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-centralizing;*
(d) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-skew commuting;*
(e) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-skew centralizing;*
(f) *$D_1 + D_2 = 0$.*

Proof. Let $\kappa \in \mathbb{N}$ and $t \in L_0^\infty(G)^*$. We obtain $(f+g)(t^\kappa) = D(t^\kappa, t^\kappa) + D(t^\kappa, t^\kappa) = f(t)(t^{2\kappa-2}) + g(t)(t^{2\kappa-2})$. Also, we obtain

$$f(t).t^\kappa = \langle f(t), t^\kappa \rangle, \qquad g(t).t^\kappa = \langle g(t), t^\kappa \rangle \tag{2.2}$$

□

Corollary 2.3. *Let $D_1, D_2 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ be symmetric bi-derivations, and let f and g be the trace of D_1 and D_2, respectively. Then the following assertions are equivalent.*

(a) *$f + g$ is (skew) centralizing;*
(b) *there exists $\kappa \in \mathbb{N}$ such that $f + g$ is κ-(skew) centralizing;*
(c) *for every $\kappa \in \mathbb{N}$, $f + g$ is κ-(skew) centralizing;*
(d) *$D_1 + D_2 = 0$.*

Corollary 2.4. *Let Let* $D_1 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ *and* $D_2 : L_0^\infty(G)^* \times L_0^\infty(G)^* \to L_0^\infty(G)^*$ *be symmetric bi-derivations,* f *and* g *be the trace of* D_1 *and* D_2*, respectively. Then the following assertions are equivalent.*

(a) $f+g$ *is commuting;*
(b) $f+g$ *is centralizing;*
(c) $f+g$ *is skew commuting;*
(d) $f+g$ *is skew centralizing;*
(e) $D_1+D_2=0$.

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