A NEW TYPE OF HADAMARD INEQUALITY AND HARMONICALLY CONVEX FUNCTIONS

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Abstract. In this paper, we establish a new type of Hadamard inequality for harmonically convex functions. Also we establish several results for convex and harmonically convex functions. In fact, we obtain some results for sum, difference, composition and absolute value of these classes of functions.

1. INTRODUCTION AND PRELIMINARIES

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds, [2]. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) by appropriate selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize,

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improve and extend the inequalities (1.1) we refer the reader to the recent papers, see [1,3].

The main purpose of this paper is to introduce the concept of harmonically convex functions (briefly HCF) and establish some results connected with these classes of functions. In the end we obtained some results for sum, difference, composition and absolute value of HCF.

Definition 1.1. Let $I \subset \mathbb{R} - \{0\}$ be a real interval. A function $f : I \longrightarrow \mathbb{R}$ is said to be harmonically convex if

$$f(\frac{xy}{tx + (1-t)y}) \leq tf(y) + (1-t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0, 1]$.

If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

Example 1.2. Let $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = cx$, $c > 0$ then f is an HCF.

Proof. Because for all $x, y \in (0, \infty)$ and $t \in [0, 1]$, we have $(x - y)^2 \geq 0$ and $t(x-y)^2 - t^2(x-y)^2 \geq 0$, thus

$$c(tx^2 - 2txy + ty^2 - t^2x^2 + 2t^2xy - t^2y^2 + xy) \geq cxy$$

and

$$c(tx + y - ty)(ty + x - tx) \geq cxy.$$

Since $x > 0, y > 0$ and $tx + (1-t)y \neq 0$, $\frac{cxy}{tx+(1-t)y} \leq c(ty + (1-t)x)$. Therefore we have

$$f(\frac{xy}{tx + (1-t)y}) \leq tf(y) + (1-t)f(x).$$

□

In [2], the author gave the definition of HCF and established some Hermite-Hadamard type inequalities for harmonically convex functions as follows.

Theorem 1.3 (Theorem 1 in [2])**.** *Let $f : I \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be an* HCF *and $a, b \in I$ with $a < b$. If f is integrable on $[a, b]$ then the following inequalities hold*

$$f(\frac{2ab}{a+b}) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a) + f(b)}{2} \tag{1.3}$$

2. Main Results

The main purpose of this section is to establish some results about HCF.

Theorem 2.1. *Let* $f : I \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ *be an HCF and* $a, b \in I$ *with* $a < b$. *If* f *is integrable on* $[a, b]$ *then*

$$f(\frac{nab}{\lambda_1 a + \lambda_2 b}) \leq \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \leq \frac{mf(b) + (n-m)f(a)}{n} \tag{2.1}$$

where

$$\lambda_1 = 2m(1 - \frac{m}{n}), \quad \lambda_2 = n - 2m(1 - \frac{m}{n}), \quad m, n \in \mathbb{N}$$

Proof. Since $f : I \longrightarrow \mathbb{R}$ is an HCF, for all $x, y \in I$ (with $0 < t = m/n < 1$ in the inequality (1.2)) we have

$$f(\frac{nxy}{mx + (n-m)y}) \leq \frac{mf(y) + (n-m)f(x)}{n}.$$

Choosing $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{tb+(1-t)a}$, we get

$$f(\frac{nab}{\lambda_1 a + \lambda_2 b}) \leq \frac{m}{n} f(\frac{ab}{tb + (1-t)a}) + \frac{n-m}{n} f(\frac{ab}{ta + (1-t)b}).$$

Further, by integrating for $t \in [0, 1]$, we have

$$\begin{aligned} f(\frac{nab}{\lambda_1 a + \lambda_2 b}) \leq \frac{1}{n}[m \int_0^1 f(\frac{ab}{tb + (1-t)a})dt \\ + (n-m) \int_0^1 f(\frac{ab}{ta + (1-t)b})dt] \end{aligned} \tag{2.2}$$

Making the change of variables

$$\frac{ab}{tb + (1-t)a} = u, \qquad \frac{-(b-a)ab}{(tb + (1-t)a)^2}dt = du$$

in last integrals in (2.2), we have

$$\int_0^1 f(\frac{ab}{tb + (1-t)a})dt = \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du. \tag{2.3}$$

From (2.2) and (2.3), we have

$$f(\frac{nab}{\lambda_1 a + \lambda_2 b}) \leq \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \tag{2.4}$$

For the second inequality, use (1.2) with $x = a$,$y = b$ and get

$$f(\frac{ab}{ta + (1-t)b}) \leq \frac{mf(b) + (n-m)f(a)}{n} \tag{2.5}$$

Now, by integrating with respect to t over $[0, 1]$ from (2.5), we have

$$\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \leq \frac{mf(b) + (n-m)f(a)}{n}$$

This completes the proof. □

Proposition 2.2. (i) *Let* $f, g : I \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ *be* HCF *and* $D_f \cap D_g \neq \varnothing$. *Then* $f + g$,$f - g$ *are* HCF.

(ii) *Let* $f : I \subset \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ *be a* HCF, g *be a nondecreasing and convex function such that* $R_f \cap D_g \neq \varnothing$. *Then* $g \circ f$ *is* HCF.

Example 2.3. Let $g(x) = e^x$ and f be an HCF. Then it is clear that $(g \circ f)(x) = e^{f(x)}$ is HCF.

References

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