φ-Splitting of Short Exact Sequences and φ-Connes Amenability of Some Banach Algebras

Ebrahim tamimi

Department of Mathematics and Engineering Science, Velayat University, Iranshahr; e.tamimi@velayat.ac.ir

Abstract

Let φ and ψ be w*-continuous homomorphisms from dual Banach algebras A and B to $\mathbb{C}$. In this paper, we present some characterizations of φ-Connes amenability and ψ-Connes amenability of dual Banach algebra A and B with preduals A_* and B_*, respectively in terms of so-called φ-splitting and ψ-splitting of the short exact sequences. Also, we investigate the relation between φ-splitting of the certain short exact sequence and φ-σwc virtual diagonal of a Banach algebra. The relation between φ-splitting and ψ-splitting with $\varphi\otimes\psi$-splitting of the certain short exact sequence is obtained. We know that the admissible and splitting short exact sequences play important role in this talk. Also, we investigate the relation between φ- σwc virtual diagonals and the mentioned notions. We obtain the relation between φ-Connes amenability of A, φ-splitting of the certain short exact sequence and φ- σwc virtual diagonals. Other results in this direction are also obtained.

Keywords: φ-σwc virtual diagonal, φ-Connes amenability, φ-splitting, dual Banach algebra, short exact sequence

1. Introduction

Connes amenability of certain Banach algebras in terms of normal virtual diagonals is characterized by Effros in [3]. Ghaffari and Javadi in [4], investigated φ-Connes amenability for dual Banach algebras and semigroup algebras, where φ was an homomorphism from a Banach algebra on $\mathbb{C}$. In [7], Runde proved that the measure algebra M(G) for a locally compact group G is Connes amenable if and only if it has a normal virtual diagonal if and only if G is amenable. Also in [5], Ghaffari et al. investigated φ-Connes module amenability of dual Banach algebras that φ is a w^*-continuous bounded module homomorphism from a Banach algebra on itself. In [2, pro 4.4], Daws proved that a Banach algebra is Connes amenable if and only if the short exact sequence splits. What is the relation between φ-splitting and φ-Connes amenability, where φ is w^*-continuous homomorphism from Banach algebra onto $\mathbb{C}$. Motivated by above question and [8, 9], to study φ-Connes amenability and φ-splitting. In fact, we obtain a characterization for φ-Connes amenability of a dual Banach algebra $A = (A_*)^*$, in terms of so-called φ-splitting of the short exact sequences. Strong Connes amenability of certain dual Banach algebras is studied in [12] by Tamimi and Ghaffari. Also, dual of group algebras under a locally convex topology is investigated by Ghaffari et al. [11].

In this paper we investigate the relation between φ-splitting and φ- σwc virtual diagonals of Banach algebras. Also, the relation between two short exact sequences $\sum_\varphi$, and $\sum_\psi$, with $\sum_{\varphi\otimes\psi}$, that are φ, ψ and $\varphi\otimes\psi$-splitting, respectively is investigated. The equivalence relation between $\varphi\otimes\psi$- σwc virtual diagonals and $\varphi\otimes\psi$-Connes amenability of projection tensor product of Banach algebras is obtained. The biflat of a Banach algebra under some natural conditions, is investigated.

In finally, we obtain a condition for dual Banach algebra under which, the short exact sequence φ-splits.

We recall that for Banach algebra A, the projective tensor product $A\widehat{\otimes}A$ is a Banach A-bimodule in the canonical way. Now, we define the map A-bimodule homomorphism $\pi : A\widehat{\otimes}A \to A$ by $\pi(a\otimes b) = ab$. A Banach A-bimodule E is dual if there is a closed submodule $E_* \subseteq E^*$, predual of E, such that $E = (E_*)^*$. A dual Banach A-bimodule E is normal if the module actions of A on E are w*-continuous. A Banach algebra is dual if it is dual as a Banach A-bimodule. We write $A = (A_*)^*$. Let $A = (A_*)^*$ be a dual Banach algebra and let E be a Banach A-bimodule. Then σwc(E), a closed submodule of E, stands for the set of all elements $x \in E$ such that the following maps are w^*- w continuous

$$A \to E; \quad a \to a.x, \quad a \to x.a.$$

The Banach A-bimodules E that are relevant to us are those the left action is of the form $a.x = \varphi(a)x$. For the brevity's sake, such E will occasionally be called a Banach φ-bimodule.

Throughout the paper, $\Delta(A)$ and $\Delta_{w^*}(A)$ will denote the sets of all homomorphisms and w^*-continuous homomorphisms from the Banach algebra A onto $\mathbb{C}$, respectively.

In [3], the Connes amenability of certain Banach algebras in terms of normal virtual diagonals is characterized by Effros. Ghaffari and Javadi in [4], investigated φ -Connes amenability for dual Banach algebras, where φ is an homomorphism from a Banach algebra on $\mathbb{C}$. Recently, in [10], Ghaffari et al. investigated ψ-Connes module amenability of dual Banach algebras that ψ is a w^*-continuous bounded module homomorphism from a Banach algebra on itself. In [2, pro 4.4], the author proved that a Banach algebra is Connes amenable if and only if the short exact sequence splits. In [1], the concept of module amenability for Banach algebras is introduced. In this talk, we study the relation between φ-splitting and φ-Connes module amenability, where φ is a w^*-continuous bounded module homomorphism from Banach algebra A. In fact, we give a characterization of φ-Connes module amenability of a dual Banach algebra in terms of so-called φ-splitting of the certain short exact sequences

A Banach A-bimodule E is dual if there is a closed submodule $E_* \subseteq E^*$ such that $E = (E_*)^*$. We say E_* predual of E. Throughout the talk, $\Delta(A)$ and $\Delta_{w^*}(A)$ will denote the sets of all homomorphisms and w^*-continuous homomorphisms from the Banach algebra A onto $\mathbb{C}$, respectively.

2. Short exact sequences and φ-splitting

Let A be a Banach algebra, and let E be a Banach A-bimodule. A derivation from A to E is a bounded, linear map $D : A \to E$ satisfying $D(ab) = a.D(b) + D(a).b$ $(a, b \in A)$. A derivation $D : A \to E$ is called inner if there is $x \in E$ such that $Da = a.x - x.a$ $(a \in A)$.

Definition 2.1. Let A be a Banach algebra, and let $3 \leq n$ $n \in \mathbb{N}$. A sequence $A_1 \to A_2 \to \ldots \to A_n$ of A-bimodules $A_1, A_2, \ldots, A_n$ and A-bimodule homomorphisms $\varphi_i : A_i \to A_{i+1}$ for $i = 2, \ldots, n-1$ is called exact at position $i = 2, \ldots, n-1$ if $\varphi_{i-1} = \ker \varphi_i$. It is called exact if it is exact at every position $i = 2, \ldots, n-1$.

We restrict ourselves to exact sequences with few bimodules, and a few bimodules (short exact sequences) respectively. Therefore, an exact sequence of the following form $0 \to A_1 \to A_2 \to A_3 \to 0$ is called a short exact sequence.

In the following we define the admissible and splitting short exact sequences.

Definition 2.2. Let A be a Banach algebra. A short exact sequence

$$\theta : 0 \to A_1 \to A_2 \to \ldots \to A_n \to 0$$

of Banach A-bimodules $A_1, A_2, \ldots, A_n$ and A-bimodule homomorphisms $\varphi_i : A_i \to A_{i+1}$ for $i = 1, 2, \ldots n-1$ is admissible, if there exists a bounded linear map $\rho_i : A_{i+1} \to A_i$ such that $\rho_i \circ \varphi_i$ on A_i for $i = 1, 2, \ldots n-1$ is the identity map. Further, θ splits if we may choose ρ_i to be an A-bimodule homomorphism.

Let $A = (A_*)^*$ be an unital dual Banach algebra by unit of $\mathbf{e}$. Then the short exact sequence

$$\textstyle\sum: 0 \to A_* \to \sigma wc\,(A\hat{\otimes}A)^* \to \sigma wc\,\frac{(A\hat{\otimes}A)^*}{\pi^*(A_*)} \to 0,$$

of A-bimodules is admissible (indeed, the map $\rho : \sigma wc\,(A\hat{\otimes}A)^* \to A_*$ defined by $\rho(T) = T(\mathbf{e})$ is a bounded left inverse to $\pi^*|A_*$). We restrict ourselves to the case where $\varphi \in \Delta_{w^*}(A) \cap A_*$.

Definition 2.3. Let $A = (A_*)^*$be an unital dual Banach algebra, and let $\varphi \in \Delta_{w^*}(A) \cap A_*$. We say that $\sum_\varphi$ φ-splits if there exists a bounded linear map $\rho: \sigma wc\,(A\hat{\otimes}A)^* \to A_*$ such that $\rho \circ \pi^*(\varphi) = \varphi$ and $\rho(T.a) = \varphi(a)\rho(T)$, for all $a \in A$ and $T \in \sigma wc\,(A\hat{\otimes}A)^*$.

Example 2.4. (i) Let A be an unital Banach algebra. The short exact sequence of Banach A-bimodules, $0 \to \ker \pi \to A\hat{\otimes}A \to A \to 0$, is admissible.

(ii) Let $A = (A_*)^*$be an unital dual Banach algebra. Then the short exact sequence

$$\textstyle\sum_\varphi: 0 \to A_* \to \sigma wc\,(A\hat{\otimes}A)^* \to \sigma wc\,\frac{(A\hat{\otimes}A)^*}{\pi^*{}_A(A_*)} \to 0,$$

of A-bimodules is admissible.

Definition 2.5. Let A be a dual Banach algebra, and let $\varphi \in \Delta_{w^*}(A) \cap A_*$. An element $M \in \sigma wc\,((A\hat{\otimes}A)^*)^*$ is a φ-σwc virtual diagonal for A if

(i) $a.M = \varphi(a)M$, $(a \in A)$;

(ii) $< \phi \otimes \phi, M> = 1$.

In throughout this paper, let $\otimes_\omega$ stand for the injective tensor product of Banach algebras.

We consider the following short exact sequences, which have three non-zero terms:

$$\textstyle\sum_\varphi: 0 \to A_* \to \sigma wc\,(A\hat{\otimes}A)^* \to \sigma wc\,\frac{(A\hat{\otimes}A)^*}{\pi^*{}_A(A_*)} \to 0,$$

$$\textstyle\sum_\psi: 0 \to B_* \to \sigma wc\,(B\hat{\otimes}B)^* \to \sigma wc\,\frac{(B\hat{\otimes}B)^*}{\pi^*{}_B(B_*)} \to 0$$

and

$$\textstyle\sum_{\varphi\otimes\psi}: 0 \to A_*\otimes_\omega B_* \to \sigma wc\,((A\hat{\otimes}B)\hat{\otimes}(A\hat{\otimes}B))^* \to \sigma wc\,((A\hat{\otimes}B)\hat{\otimes}(A\hat{\otimes}B))^*/\pi^*{}_{A\hat{\otimes}B}(A_*\otimes_\omega B_*) \to 0$$

Definition 2.7. Let A be a Banach algebra and $\pi : A\hat{\otimes}A \to A$ is the projection induced product map. Then A is biflat if $\pi^* : A^* \to (A\hat{\otimes}A)^*$ has a bounded left inverse which is an A-bimodule homomorphism.

In following we extend Daws's theorem under certain condition on a Banach algebra [2].

Theorem 2.8. Let $A = (A_*)^*$ be an unital biflat dual Banach algebra, and let $\varphi \in \Delta_{w^*}(A) \cap A_*$. Then the following are equivalent:

(i) the short exact sequence $\sum_\varphi$ φ-splits.

(ii) there is a φ- σwc virtual diagonal for A.

Theorem 2.9. *Suppose that* $A = (A_*)^*$, $B = (B_*)^*$ *and* $A\widehat{\otimes}B = (A_*\otimes_\omega B_*)^*$ *be unital dual Banach algebras, and let* $\varphi \in \Delta_{w^*}(A)\cap A_*$ *and* $\psi \in \Delta_{w^*}(B)\cap B_*$. *If the short exact sequences* $\sum_\varphi \varphi$ *and* $\sum_\psi \psi$ *are* φ*-splits and* ψ*-splits, respectively. Then the short exact sequence* $\sum_{\varphi\otimes\psi} \varphi\otimes\psi$ *-splits.*

3. φ-Connes amenability and φ-splitting

Connes amenability for certain product of Banach algebras play important role in cohomology notions of Banach algebras [6]. In this section our main aim is investigation of the relation between notions of φ-Connes amenability of dual Banach algebras and φ-splitting of the short exact sequences. In this section we discuss some hereditary properties of φ-Connes amenability. The main result concerns the projective tensor product of two dual Banach algebras.

Definition 3.1. ([4, Definition 2.1]) Let A be a dual Banach algebra and $\varphi \in \Delta_{w^*}(A)\cap A_*$. A is φ-Connes amenable if for every normal φ-bimodule E, every bounded w*-continuous derivation $D : A \to E$ is inner.

Definition 3.2. ([4, Definition 2.2]) Let A be a dual Arens regular Banach algebra with predual A_* and $\varphi \in \Delta_{w^*}(A)\cap A_*$. A linear functional $m \in A^{**}$ is called a mean if $m(\varphi) = 1$ and m is called a φ invariant mean if $m(a.f) = \varphi(a)\, m(f)$ for all $a \in A$ and $f \in A_*$.

Proposition 3.3. *Suppose that* $A = (A_*)^*$, $B = (B_*)^*$ *be unital biflat dual Banach algebras and* $A\widehat{\otimes}B = (A_*\otimes_\omega B_*)^*$ *be a Banach algebra. Let* $\varphi \in \Delta_{w^*}(A)\cap A_*$ *and* $\psi \in \Delta_{w^*}(B)\cap B_*$. *If the short exact sequence* $\sum_{\varphi\otimes\psi} \varphi\otimes\psi$ *-splits, then the short exact sequences* $\sum_\varphi \varphi$ *and* $\sum_\psi \psi$ *are* φ*-splits and* ψ*-splits, respectively.*

Theorem 3.4. *Let* $A = (A_*)^*$ *be an unital dual Banach algebra, and let* $\varphi \in \Delta_{w^*}(A)\cap A_*$. *If the short exact sequence* $\sum_\varphi$ φ*-splits, then* $\{c \in A : \varphi(c) = 0\}$ *has a left identity.*

Corollary 3.5. *Let* A *be a dual Banach algebras, let* $\varphi \in \Delta_{w^*}(A)\cap A_*$. *Then the following are equivalent:*

(*i*) *the short exact sequence* $\sum_\varphi$ φ*-splits;*

(*ii*) *there is a* φ*- σwc virtual diagonal for* A*;*

(*iii*) A *is* φ*-Connes amenable.*

Corollary 3.6. *Let* A *and* B *be unital biflat dual Banach algebras, let* $\varphi \in \Delta_{w^*}(A)\cap A_*$ *and* $\psi \in \Delta_{w^*}(B)\cap B_*$. *Then the following are equivalent:*

(*i*) $A\widehat{\otimes}B$ *is* $\varphi\otimes\psi$ *-Connes amenable;*

(*ii*) *there is a* $\varphi\otimes\psi$*- σwc virtual diagonal for* $A\widehat{\otimes}B$*;*

(*iii*) *the short exact sequence* $\sum_{\varphi\otimes\psi} \varphi\otimes\psi$ *-splits.*

Corollary 3.7. *Let* $A = (A_*)^*$ *be a dual Arens regular Banach algebra, and let* $\varphi \in \Delta_{w^*}(A)\cap A_*$. *If* A^{**} *has a* φ*-invariant mean, then the short exact sequence* $\sum_\varphi$ φ*-splits.*

4. Conclusions

In present paper, we studied some characterizations of φ-Connes amenability and ψ-Connes amenability of dual Banach algebra A and B with preduals A_* and B_*, respectively in terms of so-called φ-splitting and ψ-splitting of the short exact sequences, where *φ and ψ are two homomorphisms on* A and B, respectively. Also, we investigate the relation between φ-splitting of the certain short exact sequence and *φ-σwc* virtual diagonal of a Banach algebra. The relation between φ-splitting and ψ-splitting with $\varphi\otimes\psi$-splitting of the certain short exact sequence is obtained.

References

[1] Amini, M ., 2004. "Module amenability for semigroup algebras". *Semigroup Forum*, 69, pp. 243–254.

[2] Daws, M., 2006. "Connes amenability of bidual and weighted semigroup algebras". *Math. Scand*, 99, pp. 217–246.

[3] Effros, E. G., 1988. "Amenability and virtual diagonals for Von Neumann algebras". *J. Funct. Anal*, 78, pp. 137–156.

[4] Ghaffari, A., and Javadi, S., 2017. "φ-Connes amenability of dual Banach algebras". *Bull. Iranian Math. Soc*, 43, pp. 25–39.

[5] Ghaffari, A., Javadi, S., and Tamimi, E., 2020. "*φ*-Connes module amenability of dual Banach algebras". *Journal of Algebraic Systems*, 8(1), pp. 69-82.

[6] Ghaffari, A., Javadi, S., and Tamimi, E., 2022. "Connes Amenability for Certain Product of Banach Algebras". *Wavelets and Linear Algebra*, 9(1), pp. 1-14.

[7] Runde, V., 2003. "Connes amenability and normal virtual diagonals for measure algebras II". *Bull. Aust. Math. Soc.*, 68, pp. 325-328.

[8] Runde, V., 2002. *Lectures on Amenability*, *Lecture Notes in Mathematics*, Springer Verlag, Berlin.

[9] Runde, V., 2004. "Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and

injectivity of the predual bimodule". *Math. Scand*, *95*, *PP*. 124–144.

[10] Tamimi, E., and Ghaffari, A., 2023. "A Survey on χ-module Connes amenability of semigroup algebras". *Journal of Algebraic Systems*, 29 May, Accepted to Online Publish. https://doi.org/10.21203/rs.3.rs-2676837/v1

[11] Ghaffari, A., Sheibani, M., and Tamimi, E., 2023. "On dual of group algebras under a locally convex topology". *journal of Advanced Mathematical Modeling (JAMM)*, 13(2), pp. 307–315. https://doi.org/10.22055/JAMM.2023.43770.2164

[12] Tamimi, E., and Ghaffari, A., 2024. "On $\chi \otimes \eta$-strong Connes amenability of certain dual Banach algebras ". *J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math.* 31(1), Accepted for publish in Feb 2024.