

On Module Connes Amenability for $A \times_{\theta} B$ and $A \oplus \Omega$ Algebras

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Abstract

Let $A = (A^*)^*$ and $B = (B^*)^*$ be two Banach algebras where A^* and B^* are preduals of A and B respectively. Let Ω be a normal Banach A -bimodule with predual Ω^* and $\Theta : B \rightarrow A$ be an algebraic homomorphism. In this paper, we investigate the module Connes amenability for Θ -Lau product of Banach algebras. Also, we investigate the conditions for module Connes amenability of Θ -Lau product of Banach algebras. Moreover, we characterize the (ψ, Θ) -module Connes amenability of Θ -Lau product $A \times_{\Theta} B$. Finally, we obtain characterization for $(\psi, 0)$ -module Connes amenability of module extension of Banach algebra $A \oplus \Omega$.

Keywords: module extension Banach algebra, φ -module Connes amenable, module φ -derivation, Θ -Lau product.

1. Introduction

Johnson in [11], established the source of term 'amenability' for Banach algebras. The concept of amenability of a Banach algebra A to the case that there exists an additional U -module structure on A , where U is Banach algebra is extended by Amini in [1]. Also, Amini investigate equivalence conditions for module amenability in terms of a special net and an element, called module approximate diagonal and module virtual diagonal respectively. The definition of Connes amenability makes sense for a larger class of Banach algebras. Runde in [19, 20, 21, 22], introduced the concept of Connes amenability of dual Banach algebras and this type of Banach algebras is interested by many researchers. The concept of module Connes amenability for dual Banach algebras which are also Banach modules with respect to the compatible action, first defined by Amini in [2]. It is shown that there exists a relation between module Connes amenability and existence a normal module virtual diagonal. Also, weak amenability of module extension of Banach algebra has further investigated and deepened by Zhang [23].

The θ -Lau product of $A \times_{\theta} B$ where A and B are two Banach algebras and θ is a nonzero multiplicative functional on B was introduced by Lau in [14] for certain class of Banach algebras and followed by Monfared in [16] for the general case.

Let E be a Banach A -bimodule. The collection of all elements of E that module maps from A onto E are w^* -weakly continuous, is denoted by $\sigma_{wc}(E)$. A σ_{wc} -virtual

diagonal for A is an element $\mu \in \sigma_{wc}((A \hat{\otimes} A)^*)$ such that $a \cdot \mu = \mu \cdot a$ and $a \cdot \Delta_{\sigma_{wc}} = a$ where, $\Delta : A \hat{\otimes} A \rightarrow A$ is the multiplication operator. In [21], it is shown that a Banach algebra A is Connes amenable if and only if it has a so-called σ_{wc} -virtual diagonal. Let A be a Banach algebra, A -bimodule E is called normal if it is a dual space such that the module actions are separately w^* -continuous [10, 11]. Recently the authors have introduced the new version, as called φ -Connes amenability of dual Banach algebra A that $\varphi \in \Delta(A)$, the set of all continuous homomorphisms from A onto Complex numbers \mathbb{C} , and also $\varphi \in A^*$ [8], as follows:

A dual Banach algebra A is φ -Connes amenable if, for every normal A -bimodule E , where the left action is of the form $a \cdot e = \varphi(a)e$; ($a \in A$, $e \in E$), every bounded w^* -continuous derivation $D : A \rightarrow E$ is inner.

In this paper, we are going to investigate the module Connes amenability and character module Connes amenability for $A \times_{\theta} B$. Suppose that A is a Banach algebra, F is a subspace of A and $\varphi \in \Delta(A) \cap F$.

A linear functional \mathbf{m} on F is said to be a mean if

$$\langle \mathbf{m}, \varphi \rangle = 1.$$

A mean \mathbf{m} is φ -invariant mean if

$$\langle \mathbf{m}, a \cdot f \rangle = \varphi(a) \langle \mathbf{m}, f \rangle$$

for all $a \in A$ and $f \in F$.

Note that φ -Connes amenability of A follows from $F = A^*$ (see [12, 9, 8]).

The authors showed that a dual Banach algebra A is φ -Connes amenable if and only if the second dual, A^{**} has a φ -invariant mean on A^* .

Consider A is a dual Banach algebra, and U is a Banach algebra such that A is a Banach U -bimodule via following actions,

$$(1) v \cdot (ab) = (v \cdot a) \cdot b$$

$$(2) (v \cdot w) \cdot a = v \cdot (w \cdot a)$$

for every $a, b \in A$ and $v, w \in U$.

Let E be a normal dual Banach A -bimodule. Moreover, if E is an U -bimodule via

$$(1) v \cdot (a \cdot e) = (v \cdot a) \cdot e$$

$$(2) (a \cdot v) \cdot e = a \cdot (v \cdot e)$$

$$(3) (v \cdot e) \cdot a = v \cdot (e \cdot a)$$

for every $a \in A$, $v \in U$ and $e \in E$, then we say that E is a normal Banach left A - U -bimodule. Similarly, for the right



actions. Also, we say that E is symmetric, if $v \cdot e = e \cdot v$ ($v \in U, e \in E$).

Let U and A be Banach algebras, and let E be a Banach A - U -bimodule. Then By using [18], E^* becomes a Banach U - A -bimodule via

$$\langle e, a \cdot f \rangle := \langle e \cdot a, f \rangle, \langle e, f \cdot v \rangle := \langle v \cdot e, f \rangle$$

For every $v \in U, a \in A, f \in E^*$ and $e \in E$.

Definition 1.1. Let A be a Banach algebra, U be a Banach algebra such that A is a Banach U -bimodule, $\varphi \in \Delta(A) \cap A^*$ and E be a Banach A - U -bimodule. A bounded map D_U from A to E is called a module φ -derivation if

$$(1) D_U(v \cdot a \pm b \cdot w) = v \cdot D_U(a) \pm D_U(b) \cdot w;$$

$$(2) D_U(ab) = D_U(a) \cdot \varphi(b) + \varphi(a) \cdot D_U(b)$$

for all $a, b \in A$ and $v, w \in U$.

Definition 1.2. Let A be a Banach algebra, U be a Banach algebra such that A is a Banach U -bimodule and $\varphi \in \Delta(A) \cap A^*$. We say that A is φ -module Connes amenable if for any symmetric normal Banach A - U -bimodule E , each w^* -continuous module φ -derivation $D_U : A \rightarrow E$ is inner.

Module amenability is somehow weaker than amenability. Suppose that A is a Banach U -bimodule and φ is an identity map on Banach algebra A , then id -module Connes amenability of A is equivalent to module Connes amenability of A . Also, by the similar argument of the proof of [1, Proposition 2.1], above notations and that U has a bounded approximate identity for A , it can easily be seen that Connes amenability of A follows its module Connes amenability.

Remark 1.3. In [9], it is defined a certain version of φ -module Connes amenability where $\varphi: A \rightarrow A$ is a module homomorphism, as follows:

Let A be a Banach algebra, U be a Banach algebra such that A is a Banach U -bimodule and $\varphi : A \rightarrow A$ is a map that satisfied

$$\varphi(v \cdot a + b \cdot w) = v \cdot \varphi(a) + \varphi(b) \cdot w, \quad \varphi(ab) = \varphi(a)\varphi(b),$$

for every $a, b \in A; v, w \in U$.

In this case, A is called φ -module Connes amenable if for any symmetric normal Banach A - U -bimodule E , each w^* -continuous module φ -derivation from A to E is inner [9].

2. (Ψ, Θ) -module Connes amenability of Θ -Lau $A \times_{\Theta} B$

Suppose that A is an unital dual Banach algebra with predual A^* , the identity e_A and B is a dual Banach algebra with predual B^* . Suppose that $\psi \in \Delta(A) \cap A^*, \theta \in \Delta(B) \cap B^*$ and $\Theta: B \rightarrow A$ is an algebraic homomorphism, which $\Theta(b) = \theta(b) e_A$.

The Θ -Lau product $A \times_{\Theta} B$ is defined with

$$(a, b) \cdot (a', b') = (a \cdot a' + \Theta(b') \cdot a + \Theta(b) \cdot a', bb')$$

and the norm $\| (a, b) \|_{A \times_{\Theta} B} = \| a \|_A + \| b \|_B$ for all $a, a' \in A$ and $b, b' \in B$.

This definition is a certain case of the product that is presented in [6, 14, 15, 16]. In this section, we investigate the notions of (Ψ, Θ) -module Connes amenability and $(0, \Theta)$ -module Connes amenability. Since $\theta \in \Delta(B) \cap B^*$, then $A \times_{\Theta} B$ is a dual Banach algebra with predual $A^* \times B^*$. It is known that $(A \times_{\Theta} B)^*$ identified with $A^* \times B^*$ that $\langle (f, g), (a, b) \rangle = f(a) + g(b)$ for all $a \in A, b \in B$ and $f \in A^*, g \in B^*$.

In this paper we consider that A^{**} and B^{**} are equipped with the first Arens product.

In the following we investigate module Connes amenability of $A \times_{\Theta} B$.

Lemma 2.1. Let A be an unital dual Banach algebra with predual A^* and let B be a dual Banach algebra with predual B^* . Then the following two statements are equivalent,

- (i) A and B are id -module Connes amenable.
- (ii) $A \times_{\Theta} B$ is $\text{id} \otimes \text{id}$ -module Connes amenable.

Remark 2.2. In [16], it is shown that

$$\Delta(A \times_{\Theta} B) = \{(\Psi, \theta) : \Psi \in \Delta(A)\} \cup \{(0, \varphi) : \varphi \in \Delta(B)\}$$

where, $\theta \in \Delta(B) \cap B^*$.

Now, in the following theorem we extend Lemma 3. 1.

Theorem 2.3. Let A be an unital dual Banach algebra that is Arens regular and B be a dual Banach algebra. Let $\psi \in \Delta(A) \cap A^*$ and $\theta \in \Delta(B) \cap B^*$. Then the following statements are hold:

- (i) $A \times_{\Theta} B$ is (Ψ, Θ) -module Connes amenable if and only if A is ψ -module Connes amenable.
- (ii) $A \times_{\Theta} B$ is $(0, \Theta)$ -module Connes amenable if and only if B is θ -module Connes amenable.

3. $(\Psi, 0)$ -module Connes amenability of $A \oplus \Omega$

In this section we study module Connes amenability and character module Connes amenability for module extension of dual Banach algebras. Suppose that A is a dual Banach algebra and $\Omega = (\Omega^*)^*$ is a normal Banach A -bimodule. Indeed, the Banach algebra $A \oplus \Omega$, the 1^1 -direct sum of a Banach algebra A and a nonzero Banach A -bimodule Ω , with algebraic product

$$(a, w) \cdot (a', w') = (aa', aw' + wa'); (a, a' \in A \text{ and } w, w' \in \Omega)$$

and defined norm as follows:

$$\| (a, w) \| = \| a \|_A + \| w \|_{\Omega};$$

is called module extension of Banach algebra. Some aspects of algebras of this form have been discussed in [3,

5]. It is known that $A \oplus \Omega$ is a dual Banach algebra with predual $A^* \oplus_{\infty} \Omega^*$, where \oplus_{∞} denotes l_{∞} -direct sum of Banach A -bimodules.

In [7], the authors showed that $\Delta(A \oplus \Omega) = \Delta(A) \times \{0\}$. They proved that if $A \oplus \Omega$ is $(\psi, 0)$ -amenable then A is ψ -amenable and the converse also holds in the case, where $\Omega A = 0$. Here, we extend the result for $(\psi, 0)$ -module Connes amenability. The following theorem is an analog of [13, Theorem 1.1].

Theorem 3.1. Let A be a dual Banach algebra and $\psi \in \Delta(A) \cap A^*$. Let Ω be a normal Banach A -bimodule. Then the following hold:

- (i) If $A \oplus \Omega$ is $(\psi, 0)$ -module Connes amenable, then A is ψ -module Connes amenable.
- (ii) If $\Omega A = 0$ and A is ψ -module Connes amenable, then $A \oplus \Omega$ is $(\psi, 0)$ -module Connes amenable.

Corollary 3.2. Let A be a dual Banach algebra with predual A^* and let Ω be a normal Banach A -bimodule with pseudo-unital predual Ω^* . Then the following are equivalence:

- (i) $A \oplus \Omega$ is id-module Connes amenable.
- (ii) A is id-module Connes amenable.

Conclusions

In this paper, we investigate the conditions for module Connes amenability of Θ -Lau product of Banach algebras. Moreover, we characterize the (ψ, Θ) -module Connes amenability of Θ -Lau product $A \times_{\Theta} B$. Finally, we obtain characterization for $(\psi, 0)$ -module Connes amenability of module extension of Banach algebra $A \oplus \Omega$.

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