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# Module Connes amenability for projective tensor product and Θ-Lau produt of Banach algebras

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**Abstract**

Let $\mathbb{A}$ and $\mathbb{B}$ be Banach algebras with preduals $\mathbb{A}_*$ and $\mathbb{B}_*$ respectively, and $\Theta : \mathbb{B} \to \mathbb{A}$ be an algebraic homomorphism. In this paper, we develop the notions of module Connes amenability for certain Banach algebras. Indeed, we investigate and give necessary and sufficient conditions for module Connes amenability of projective tensor product $\mathbb{A}\widehat{\otimes}\mathbb{B}$. Moreover, we characterize the $(\psi, \theta)$-module Connes amenability of Θ-Lau product $\mathbb{A} \times_{\Theta} \mathbb{B}$, which $\psi$ and $\theta$ are homomorphisms in $\mathbb{A}_*$ and $\mathbb{B}_*$, respectively.

**Keywords:** module Connes amenability, projective tensor product, Θ-Lau produt

**Mathematics Subject Classification [2023]:** 46H25, 46J10, 46L06

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## 1 Introduction and Preliminaries

The concept of amenability of a Banach algebra $\mathbb{A}$ to the case that there exists an additional $\mathbb{U}$-module structure on $\mathbb{A}$, where $\mathbb{U}$ is Banach algebra is extended by Amini. The definition of Connes amenability makes sense for a larger class of Banach algebras (called dual Banach algebras in [5]). Runde in [6], introduced the concept of Connes amenability of dual Banach algebras and this type of Banach algebras is interested by many researchers. The concept of module Connes amenability for dual Banach algebras which are also Banach modules with respect to the compatible action, first defined by Amini. It is shown that there exists a relation between module Connes amenability and existance a normal module virtual diagonal. Let $\mathbb{A}$ and $\mathbb{B}$ be two Banach algebras. The $\theta$-Lau product of $\mathbb{A} \times_\theta \mathbb{B}$ where $\theta$ is a nonzero multiplicative functional on $\mathbb{B}$ was introduced by Lau for certain class of Banach algebras and followed by Monfared in [4] for the general case.

Let $\mathbb{E}$ be a Banach $\mathbb{A}$-bimodule. The collection of all elements of $\mathbb{E}$ that module maps from $\mathbb{A}$ onto $\mathbb{E}$ are $w^*$-weakly continuous, is denoted by $\sigma wc(\mathbb{E})$. A $\sigma wc$-virtual diagonal for $\mathbb{A}$ is an element $\mu \in \sigma wc((\mathbb{A}\hat{\otimes}\mathbb{A})^*)^*$ such that $a.\mu = \mu.a$ and $a.\Delta_{\sigma wc}\mu = a$ that $\Delta : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ is the multiplication operator. In [6], it is shown that a Banach algebra $\mathbb{A}$ is Connes amenable if and only if it has a so-called $\sigma wc$-virtual diagonal. Let $\mathbb{A}$ be a Banach algebra, $\mathbb{A}$-bimodule $\mathbb{E}$ is called normal if it is a dual space such that the module actions are separately $w^*$-continuous [3].

Recently the authors have introduced the new version, as called $\varphi$-Connes amenability of dual Banach algebra $\mathbb{A}$ that $\varphi \in \Delta(\mathbb{A})$, the set of all continuous homomorphisms from $\mathbb{A}$ onto $\mathbb{C}$, and also $\varphi \in \mathbb{A}_*$ [1], as follows:

A dual Banach algebra $\mathbb{A}$ is $\varphi$-Connes amenable if, for every normal $\mathbb{A}$-bimodule $\mathbb{E}$, where the left action is of the form

$$a.\mathbf{e} = \varphi(a)\mathbf{e}, \qquad (a \in \mathbb{A}, \mathbf{e} \in \mathbb{E})$$

every bounded $w^*$-continuous derivation $\mathcal{D} : \mathbb{A} \to \mathbb{E}$ is inner.

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In this paper, we are going to investigate the module Connes amenability and character module Connes amenability for $\mathbb{A}\widehat{\otimes}\mathbb{B}$, $\Theta$-Lau product $\mathbb{A} \times_{\Theta} \mathbb{B}$.

Suppose that $\mathbb{A}$ is a Banach algebra, $\mathcal{F}$ is a subspace of $\mathbb{A}^*$ and $\varphi \in \Delta(\mathbb{A}) \cap \mathcal{F}$. A linear functional $\mathbf{m}$ on $\mathcal{F}$ is said to be a mean if $\langle \mathbf{m}, \varphi \rangle = 1$. A mean $\mathbf{m}$ is $\varphi$-invariant mean if $\langle \mathbf{m}, a.f \rangle = \varphi(a)\langle \mathbf{m}, f \rangle$ for all $a \in \mathbb{A}$ and $f \in \mathcal{F}$. Note that $\varphi$-Connes amenability of $\mathbb{A}$ follows from $\mathcal{F} = \mathbb{A}_*$ (see [1, 2]).

The authors showed that a dual Banach algebra $\mathbb{A}$ is $\varphi$-Connes amenable if and only if the second dual, $\mathbb{A}^{**}$ has a $\varphi$-invariant mean on $\mathbb{A}_*$.

Consider $\mathbb{A} = (\mathbb{A}_*)^*$ is a dual Banach algebra, and $\mathbb{U}$ is a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule via following actions,

$$v.(ab) = (v.a).b, \quad (vw).a = v.(w.a) \qquad (a, b \in \mathbb{A}, v, w \in \mathbb{U}).$$

Let $\mathbb{E}$ be a normal dual Banach $\mathbb{A}$-bimodule. Moreover, if $\mathbb{E}$ is an $\mathbb{U}$-bimodule via

$$v.(a.\mathbf{e}) = (v.a).\mathbf{e}, \ \ (a.v).\mathbf{e} = a.(v.\mathbf{e}), \ \ (v.\mathbf{e}).a = v.(\mathbf{e}.a),$$

for every $a \in \mathbb{A}$, $v \in \mathbb{U}$ and $\mathbf{e} \in \mathbb{E}$. Then we say that $\mathbb{E}$ is a normal Banach left $\mathbb{A}$-$\mathbb{U}$-bimodule. Similarly, for the right actions. Also, we say that $\mathbb{E}$ is symmetric, if $v.\mathbf{e} = \mathbf{e}.v$ $(v \in \mathbb{U}, \mathbf{e} \in \mathbb{E})$.

Let $\mathbb{U}$ and $\mathbb{A}$ be Banach algebras, and let $\mathbb{E}$ be a Banach $\mathbb{A}$-$\mathbb{U}$-bimodule. Then By using [5], $\mathbb{E}^*$ becomes a Banach $\mathbb{U}$-$\mathbb{A}$-bimodule via

$$\langle \mathbf{e}, a.f \rangle := \langle \mathbf{e}.a, f \rangle, \quad \langle \mathbf{e}, f.v \rangle := \langle v.\mathbf{e}, f \rangle, \quad (v \in \mathbb{U}, a \in \mathbb{A}, f \in E^*, \mathbf{e} \in \mathbb{E}).$$

**Definition 1.1.** Let $\mathbb{A} = (\mathbb{A}_*)^*$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule, $\varphi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$ and $\mathbb{E}$ be a Banach $\mathbb{A}$-$\mathbb{U}$-bimodule. A bounded map $\mathcal{D}_{\mathbb{U}}$ from $\mathbb{A}$ to $\mathbb{E}$ is called a *module $\varphi$-derivation* if

$$\mathcal{D}_{\mathbb{U}}(v.a \pm b.w) = v.\mathcal{D}_{\mathbb{U}}(a) \pm \mathcal{D}_{\mathbb{U}}(b).w, \quad \mathcal{D}_{\mathbb{U}}(ab) = \mathcal{D}_{\mathbb{U}}(a).\varphi(b) + \varphi(a).\mathcal{D}_{\mathbb{U}}(b),$$

for all $a, b \in \mathbb{A}$ and $v, w \in \mathbb{U}$.

**Definition 1.2.** Let $\mathbb{A}$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule and $\varphi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$. We say that $\mathbb{A}$ is *$\varphi$-module Connes amenable* if for any symmetric normal Banach $\mathbb{A}$-$\mathbb{U}$-bimodule $\mathbb{E}$, each $w^*$-continuous module $\varphi$-derivation $\mathcal{D}_{\mathbb{U}} : \mathbb{A} \to \mathbb{E}$ is inner.

**Remark 1.3.** In [2], it is defined a certain version of $\varphi$-module Connes amenability where $\varphi : \mathbb{A} \to \mathbb{A}$ is a module homomorphism, as follows:

Let $\mathbb{A}$ be a Banach algebra, $\mathbb{U}$ be a Banach algebra such that $\mathbb{A}$ is a Banach $\mathbb{U}$-bimodule and $\varphi : \mathbb{A} \to \mathbb{A}$ is a map that satisfied

$$\varphi(v.a + b.w) = v.\varphi(a) + \varphi(b).w, \quad \varphi(ab) = \varphi(a)\varphi(b),$$

for every $a, b \in \mathbb{A}, v, w \in \mathbb{U}$.

In this case, $\mathbb{A}$ is called $\varphi$-module Connes amenable if for any symmetric normal Banach $\mathbb{A}$-$\mathbb{U}$-bimodule $\mathbb{E}$, each $w^*$-continuous module $\varphi$-derivation from $\mathbb{A}$ to $\mathbb{E}$ is inner [2].

## 2 $\varphi \otimes \psi$-module Connes Amenability of projective tensor product of Banach algebras and $\varphi \otimes \psi$-invariant mean

Our aim in this section is to study of $\varphi \otimes \psi$-module Connes Amenability of projective tensor product of Banach algebras. In the sequel, in special case and by using Remark 1.3, we conclude the following result.

**Proposition 2.1.** *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are two Banach algebras. Let $\varphi : \mathbb{A} \to \mathbb{A}$ and $\psi : \mathbb{B} \to \mathbb{B}$ be two maps on Banach algebras. Suppose that $\mathbb{A}$ is $\varphi$-module Connes amenable and $\mathbb{B}$ is $\psi$-module Connes amenable. Then $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is $\eta \circ (\varphi \otimes \psi)$-module Connes amenable for any map $\eta : \mathbb{A}\widehat{\otimes}\mathbb{B} \to \mathbb{A}\widehat{\otimes}\mathbb{B}$.*

**Theorem 2.2.** *With above notations, if $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is a Banach algebra. Then $\mathbb{A}$ is $\varphi$-module Connes amenable and $\mathbb{B}$ is $\psi$-module Connes amenable if and only if $\sigma wc(\mathbb{A}\widehat{\otimes}\mathbb{B})$ is $\varphi \otimes \psi$-module Connes amenable.*

*Proof.* Let $\mathbb{A}$ be $\varphi$-module Connes amenable and $\mathbb{B}$ be $\psi$-module Connes amenable. Let $\mathbb{U}$ be a Banach algebra. Take $\mathbb{E}$ as a symmetric normal Banach $\mathbb{A}\widehat{\otimes}\mathbb{B}$-$\mathbb{U}$-bimodule and let $\mathcal{D}_{\mathbb{U}} : \mathbb{A}\widehat{\otimes}\mathbb{B} \to \mathbb{E}$ be a bounded $w^*$-continuous module $\varphi \otimes \psi$-derivation defined by

$$\mathcal{D}_{\mathbb{U}}\big((a \otimes b)(a' \otimes b')\big) = \mathcal{D}_{\mathbb{U}}(a \otimes b).(a' \otimes b') + (a \otimes b).\mathcal{D}_{\mathbb{U}}(a' \otimes b'),$$

for all $a, a' \in \mathbb{A}$ and $b, b' \in \mathbb{B}$. Hence, we obtain

$$\begin{aligned}\mathcal{D}_{\mathbb{U}}(a \otimes b) &= \varphi \otimes \psi(a \otimes b).\mathbf{y} - \mathbf{y}.(a \otimes b)\\ &= (a \otimes b).\mathbf{y} - \mathbf{y}.(a \otimes b) = (ad_{\mathbb{U}})_{\mathbf{y}}(a \otimes b),\end{aligned}$$

for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$, as required. □

**Example 2.3.** Set $\mathbb{A}=\begin{pmatrix} 0 & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$. Consider usual matrix multiplication and $l^1$-norm, $\mathbb{A}$ is a dual Banach algebra. We consider $\varphi : \mathbb{A} \to \mathbb{C}$; $\varphi\begin{pmatrix} 0 & 0 \\ z_1 & z_2 \end{pmatrix} = z_2$, for all $z_1, z_2 \in \mathbb{C}$. Note that $\varphi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$ is norm continuous and $w^*$-continuous. Let $u = \begin{pmatrix} 0 & 0 \\ 1 & -i \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ i & i \end{pmatrix} \in \mathbb{A}\widehat{\otimes}\mathbb{A}$. We can see that $u \in \sigma wc((\mathbb{A}\widehat{\otimes}\mathbb{A})^*)^*$ and therefore $u$ has the property of $\varphi - \sigma wc-$virtual diagonal for Banach algebra $\mathbb{A}$. Consequently, there exists a $\varphi \otimes \varphi$-invariant mean on $(\mathbb{A}\widehat{\otimes}\mathbb{A})^*$ and so, on $(\mathbb{A}\widehat{\otimes}\mathbb{A})_*$ (for more details see [1]).

**Corollary 2.4.** *Let $\mathbb{A}$ and $\mathbb{B}$ be Banach algebras, $\mathcal{F}$ be a subspace of $\mathbb{A}^*$ and $\mathcal{H}$ be a subspace of $\mathbb{B}^*$. Let $\varphi \in \Delta(\mathbb{A}) \cap \mathcal{F}$, $\psi \in \Delta(\mathbb{B}) \cap \mathcal{H}$, $\mathcal{F}$ be $\varphi$-module Connes amenable and $\mathcal{H}$ be $\psi$-module Connes amenable. Then $\mathcal{F}\widehat{\otimes}\mathcal{H}$ is $\varphi \otimes \psi$-module Connes amenable.*

**Theorem 2.5.** *Let $\mathbb{A} = (\mathbb{A}_*)^*$, $\mathbb{B} = (\mathbb{B}_*)^*$ and $\mathbb{A}\widehat{\otimes}\mathbb{B}$ be Banach algebras, and let $\varphi \in \Delta_{\omega^*}(\mathbb{A}) \cap \mathbb{A}_*$ and $\psi \in \Delta_{\omega^*}(\mathbb{B}) \cap \mathbb{B}_*$. Let $\mathbb{I}$, $\mathbb{J}$ be Banach algebras and closed two-sided ideals of $\mathbb{A}$, $\mathbb{B}$, respectively. If $\varphi\mid_I \neq 0$, $\psi\mid_{\mathbb{J}} \neq 0$ and $\mathbb{I}\widehat{\otimes}\mathbb{J}$ is a Banach algebra that is $\varphi\otimes\psi\mid_{\mathbb{I}\widehat{\otimes}J}$-module Connes amenable, then $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is $\varphi\otimes\psi$-module Connes amenable.*

In the sequel, under condition existence of bounded approximate identities for above ideals, $\mathbb{I}$ and $\mathbb{J}$, we show that the converse of Theorem 2.5 is hold.

**Theorem 2.6.** *Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{A}\widehat{\otimes}\mathbb{B}$, $\mathbb{I}$, $\mathbb{J}$, $\varphi$ and $\psi$ be as above. Let $\mathbb{I}$, $\mathbb{J}$ be with bounded approximate identities such that $\varphi\mid_{\mathbb{I}}$ and $\psi\mid_{\mathbb{J}}$ are non-zero and $\mathbb{I}\widehat{\otimes}\mathbb{J}$ be a Banach algebra. If $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is $\varphi\otimes\psi$-module Connes amenable, then $\mathbb{I}\widehat{\otimes}\mathbb{J}$ is $\varphi\otimes\psi\mid_{\mathbb{I}\widehat{\otimes}\mathbb{J}}$-module Connes amenable.*

## 3 $(\psi,\theta)$-module Connes amenability of $\Theta$-Lau product of Banach algebras

Suppose that $\mathbb{A}$ is an unital dual Banach algebra with predual $\mathbb{A}_*$, the identity $e_{\mathbb{A}}$ and $\mathbb{B}$ is a dual Banach algebra with predual $\mathbb{B}_*$. Suppose that $\psi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$, $\theta \in \Delta(\mathbb{B}) \cap \mathbb{B}_*$ and $\Theta : \mathbb{B} \to \mathbb{A}$ is an algebraic homomorphism, which $\Theta(b) = \theta(b)e_{\mathbb{A}}$. The $\Theta$-Lau product $\mathbb{A} \times_{\Theta} \mathbb{B}$ is defined with

$$(a, b).(a', b') = (a.a' + \Theta(b').a + \Theta(b).a', bb')$$

and the norm $\|(a, b)\|_{\mathbb{A}\times_{\Theta}\mathbb{B}} = \|a\|_{\mathbb{A}} + \|b\|_{\mathbb{B}}$ for all $a, a' \in \mathbb{A}$ and $b, b' \in \mathbb{B}$. This definition is a certain case of the product that is presented in [4]. In this section, we investigate the notions of $(\psi,\theta)$-module Connes amenability and $(0,\theta)$-module Connes amenability. Since $\theta \in \Delta(\mathbb{B}) \cap \mathbb{B}_*$, then $\mathbb{A} \times_{\Theta} \mathbb{B}$ is a dual Banach algebra with predual $\mathbb{A}_* \times \mathbb{B}_*$.

**Lemma 3.1.** *Let $\mathbb{A}$ be an unital dual Banach algebra with predual $\mathbb{A}_*$ and let $\mathbb{B}$ be a dual Banach algebra with predual $\mathcal{B}_*$. Then the following two statements are equevalent,*

1. *$\mathbb{A}$ and $\mathbb{B}$ are id-module Connes amenable.*

2. *$\mathbb{A} \times_{\Theta} \mathbb{B}$ is $id \otimes id$-module Connes amenable.*

**Theorem 3.2.** *Let $\mathbb{A} = (\mathbb{A}_*)^*$ be an unital dual Banach algebra that is Arens regular and $\mathbb{B} = (\mathbb{B}_*)^*$ be a dual Banach algebra. Let $\psi \in \Delta(\mathbb{A}) \cap \mathbb{A}_*$ and $\theta \in \Delta(\mathbb{B}) \cap \mathbb{B}_*$. Then the following statements are hold:*

1. *$\mathbb{A} \times_{\Theta} \mathbb{B}$ is $(\psi,\theta)$-module Connes amenable if and only if $\mathbb{A}$ is $\psi$-module Connes amenable.*

2. *$\mathbb{A} \times_{\Theta} \mathbb{B}$ is $(0,\theta)$-module Connes amenable if and only if $\mathbb{B}$ is $\theta$-module Connes amenable.*

## 4 Conclusion

We give necessary and sufficient conditions for module Connes amenability of $\mathbb{A}\widehat{\otimes}\mathbb{B}$ that $\mathbb{A}$ and $\mathbb{B}$ are two Banach algebras. Moreover, we characterize the $(\psi,\theta)$-module Connes amenability of $\Theta$-Lau product $\mathbb{A}\times_{\Theta}\mathbb{B}$, which $\Theta : \mathbb{B} \to \mathbb{A}$ be an algebraic homomorphism and $\psi$ and $\theta$ are homomorphisms in $\mathbb{A}_*$ and $\mathbb{B}_*$, respectively.

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