دهمین سمینار آنالیز هارمونیک و کاربردها
۱۸ و ۱۹ اسفندماه ۱۴۰۱
دانشگاه بجنورد
10th Seminar on Harmonic Analysis and Applications
9-10 March 2023
University of Bojnord

On $\phi \otimes \psi$-Strongly Connes Amenability for Certain Dual Banach Algebras

Ebrahim Tamimi *
Department of Mathematics, Velayat University, Iranshahr, Iran

ABSTRACT. In this paper, the notions of strongly Connes amenability for explicit products of Banach algebras and module extension of a Banach algebra is developed. Indeed, we characterize $\phi \otimes \psi$-strongly Connes amenability of projective tensor product $\mathbb{A}\widehat{\otimes}\mathbb{B}$ via so-called $\phi \otimes \psi$-σwc virtual diagonals and as a result $\phi \otimes \psi$-normal virtual diagonals, where $\phi \in \mathbb{A}_*$ and $\psi \in \mathbb{B}_*$, are two-linear functionals on dual Banach algebras $\mathbb{A}$ and $\mathbb{B}$, respectively. Also, we present necessary and sufficient conditions for the existence of (ϕ, θ)-σwc virtual diagonals in θ-Lau product $\mathbb{A} \times_\theta \mathbb{B}$. Finally, we characterize $(\phi, 0)$-σwc virtual diagonal and $(\phi, 0)$-strongly Connes amenability for the module extension of Banach algebra $\mathbb{A} \oplus \mathbb{X}$, where $\mathbb{X}$ is a normal Banach $\mathbb{A}$-bimodule with predual $\mathbb{X}_*$.
Keywords: Dual Banach algebra, ϕ-Strongly Connes amenability, θ-Lau product, ϕ-σwc Virtual diagonal, Module extension of Banach algebra.
AMS Mathematical Subject Classification [2010]: 46H25, 16D20, 46M10 .

1. Introduction

Let $\mathbb{A}$ be a Banach algebra, and $\mathbb{E}$ be a Banach $\mathbb{A}$-bimodule. A bounded linear map $\mathcal{D} : \mathbb{A} \to \mathbb{E}$ is a derivation if it satisfies $\mathcal{D}(\mathbf{ab}) = \mathcal{D}(\mathbf{a}).\mathbf{b} + \mathbf{a}.\mathcal{D}(\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{A}$; given $\mathbf{x} \in \mathbb{E}$, the inner derivation $ad_x : \mathbb{A} \to \mathbb{E}$ is defined by $ad_{\mathbf{x}}(\mathbf{a}) = \mathbf{a}.\mathbf{x} - \mathbf{x}.\mathbf{a}$. A Banach $\mathbb{A}$-bimodule $\mathbb{E}$ is called dual if there exists a closed submodule $\mathbb{E}_* \subseteq \mathbb{E}^*$ such that $\mathbb{E} = (\mathbb{E}_*)^*$, we say $\mathbb{E}_*$ predual of $\mathbb{E}$. The Banach algebra $\mathbb{A}$ is called dual if it is dual as a Banach $\mathbb{A}$-bimodule. We can see that a Banach algebra which is also a dual space is a dual Banach algebra if the multiplication maps are separately w^*-continuous and vice versa [9].

A dual Banach $\mathbb{A}$-bimodule $\mathbb{E}$ is normal if for every $\mathbf{x} \in \mathbb{E}$ the module maps,

$$\mathbb{A} \to \mathbb{E}, \qquad \mathbf{a} \mapsto \begin{cases} \mathbf{x}.\mathbf{a}, \\ \mathbf{a}.\mathbf{x} \end{cases} \tag{1}$$

are w^*-w^* continuous.

In [5], ϕ-strongly Connes amenability is defined and it is shown that ϕ-normal virtual diagonal and ϕ-strongly Connes amenability are equivalent.

Our main purpose is investigate and generalize of the weaker version of strongly Connes amenability for certain Banach algebras. The structure of this paper is as follow,

Firstly, we present the preliminaries in the second section. In the third section, we characterize $\phi \otimes \psi$-strongly Connes amenability of $\mathbb{A}\widehat{\otimes}\mathbb{B}$ via so-called $\phi \otimes \psi$-σwc virtual diagonals and $\phi \otimes \psi$-normal virtual diagonals, where $\phi \in \mathbb{A}_*$ and $\psi \in \mathbb{B}_*$ are two-linear functionals on dual Banach algebras $\mathbb{A}$ and $\mathbb{B}$, respectively. Also, we shall investigate (ϕ, θ)-strongly Connes amenability of θ-Lau product of $\mathbb{A} \times_\theta \mathbb{B}$ in terms of so-called (ϕ, θ)-σwc virtual diagonals, where $\theta \in \mathbb{B}_*$ is a linear functional on $\mathbb{B}$ at the end of this section. The last section is devoted to studying the $(\phi, 0)$-σwc virtual diagonals for Banach algebra of module extension $\mathbb{A} \oplus \mathbb{X}$. Also, we investigate the $(\phi, 0)$-strongly Connes amenabilty of $\mathbb{A} \oplus \mathbb{X}$, where $\mathbb{X} = (\mathbb{X}_*)^*$ be a normal Banach $\mathbb{A}$-bimodule.

2. Preliminaries and some basic Properties

DEFINITION 2.1. Let $\mathbb{A}$ be a dual Banach algebra, $\Delta_{w^*}(\mathbb{A})$ the set of all w^*-continuous homomorphisms on $\mathbb{A}$ and $\phi \in \Delta_{\omega^*}(\mathbb{A}) \cap \mathbb{A}_*$. An Banach algebra $\mathbb{A}$ is sade to be ϕ-Connes amenable if

*speaker

there exists a bounded linear functional μ on $\sigma wc(\mathbb{A})^*$ such that $\mu(\phi) = 1$ and $\mu(\mathbf{f}.\mathbf{a}) = \phi(\mathbf{a})\mu(\mathbf{f})$ for all $\mathbf{a} \in \mathbb{A}$ and $\mathbf{f} \in \sigma wc(\mathbb{A})^*$.

We apply the Banach $\mathbb{A}$-bimodules $\mathbb{E}$ whose left action is of the form $\mathbf{a}.\mathbf{x} = \phi(\mathbf{a})\mathbf{x}$, where $\phi \in \Delta_{\omega^*}(\mathbb{A}) \cap \mathbb{A}_*$. For the sake of brevity, such $\mathbb{E}$ will occasionally be called a Banach ϕ-bimodule.
Consider $\mathbb{A}$ is a dual Banach algebra, and $\mathbb{E}$ is a Banach ϕ-bimodule. Then we say $\mathbf{y} \in \mathbb{E}^*$ is a w^*-element if the map $\mathbf{a} \to \mathbf{a}.\mathbf{y}$ is w^*-continuous.

Definition 2.2. Suppose that $\mathbb{A}$ is a dual Banach algebra, $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ and $\mathbb{E}^*$ is an arbitrary Banach ϕ-bimodule. We say $\mathbb{A}$ is ϕ-strongly Connes amenable if for every w^*-continuous derivation $\mathcal{D} : \mathbb{A} \to \mathbb{E}^*$, such that $\mathcal{D}(\mathbb{A})$ consists of w^*-elements, is inner [5, Definition 2.6].

The following definition is analog to [7, Definition 3.1],

Definition 2.3. Suppose that $\mathbb{A}$ is a dual Banach algebra, and $\phi \in \Delta_{\omega^*}(\mathbb{A}) \cap \mathbb{A}_*$. $\Gamma \in \sigma wc((\mathbb{A}\widehat{\otimes}\mathbb{A})^*)^*$ is a ϕ-σwc virtual diagonal for $\mathbb{A}$ if $\langle \phi \otimes \phi, \Gamma\rangle = 1$ and $\mathbf{a}.\Gamma = \phi(\mathbf{a}).\Gamma$ for all $\mathbf{a} \in \mathbb{A}$.

Remark 2.4. By [7, Theorem 3.2], it is easy to check that ϕ-Connes amenability of a Banach algebra $\mathbb{A}$ is equivalent to existence the ϕ-σwc virtual diagonal for $\mathbb{A}$.

With above-arrangement, we characterize the ϕ-strongly Connes amenability via the existence of above defined diagonals. Indeed, we give a technical lemma to reach our purpose.

Lemma 2.5. *Suppose that $\mathbb{A}$ is a dual Banach algebra, and $\phi \in \Delta_{\omega^*}(\mathbb{A}) \cap \mathbb{A}_*$. Then $\mathbb{A}$ is ϕ-strongly Connes amenable if $\mathbb{A}$ has a ϕ-σwc virtual diagonal and vice versa.*

We apply $\otimes_w$ for the injective tensor product of two Banach spaces and suppose that $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is a dual Banach algebra with predual $\mathbb{A}_*\otimes_\omega\mathbb{B}_*$. As an easy result of the previous theorem we have the following proposition.

Proposition 2.6. *Let $\mathbb{A} = (\mathbb{A}_*)^*$, $\mathbb{B} = (\mathbb{B}_*)^*$ and $(\mathbb{A}_*\otimes_\omega\mathbb{B}_*)^*$ be dual Banach algebras. Let $\phi \in \Delta_{\omega^*}(\mathbb{A}) \cap \mathbb{A}_*$, and $\psi \in \Delta_{\omega^*}(\mathbb{B}) \cap \mathbb{B}_*$. If $\mathbb{A}$ has a ϕ-σwc virtual diagonal and $\mathbb{B}$ has a ψ-σwc virtual diagonal, then $(\mathbb{A}_*\otimes_\omega\mathbb{B}_*)^*$ has a $\phi\otimes\psi$-σwc virtual diagonal.*

Proof. By [5, Theorem 3.1], this is immediate verifications and so, the proof is complete. □

3. $\phi \otimes \psi$-strongly Connes amenability and (ϕ, θ)-σwc virtual diagonals

The purpose of this section is to investigate the $\phi \otimes \psi$-strongly Connes amenability of projective tensor product of Banach algebras and characterization θ-Lau product of Banach algebras.

3.1. $\phi \otimes \psi$-strongly Connes amenability and some hereditary properties. In this subsection, we investigate the relation among $\phi \otimes \psi$-strongly Connes amenability of projective tensor product of Banach algebras, ϕ-σwc virtual diagonals and ψ-σwc virtual diagonals. Also, the existence of $\phi \otimes \psi$-normal virtual diagonals is investigated for projective tensor product of Banach algebras. We now apply the proof of Lemma 2.5 with appropriate adjustments in the module actions and the derivations for the following theorem.

Theorem 3.1. *Suppose that $\mathbb{A}$ and $\mathbb{B}$ are unital dual Banach algebras, $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ and $\psi \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$. Then $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is $\phi \otimes \psi$-strongly Connes amenable if and only if $\mathbb{A}$ has a ϕ-σwc virtual diagonal and $\mathbb{B}$ has a ψ-σwc virtual diagonal.*

Indeed, in Theorem 3.1, we characterize the $\phi \otimes \psi$-strongly Connes amenability of projective tensor product Banach algebras via ϕ-σwc virtual diagonals and ψ-σwc virtual diagonals.
Theorem 3.1, allows us to prove the following poroposition.

Proposition 3.2. *With above notations, $\mathbb{A}\widehat{\otimes}\mathbb{B}$ is $\phi \otimes \psi$-strongly Connes amenable if $\mathbb{A}$ is ϕ-strongly Connes amenable, and $\mathcal{B}$ is ψ-strongly Connes amenable and vice versa.*

We know that in [2] and [4], the Connes amenability for some Banach algebras is characterized via normal virtual diagonals. We generalize this subject to $\phi \otimes \psi$-strongly Connes amenability, ϕ-normal virtual diagonals and ψ-normal virtual diagonals.

REMARK 3.3. In [5, Theorem 2.7], it is shown that ϕ-normal virtual diagonal and ϕ-strongly Connes amenability are equivalent.

DEFINITION 3.4. Let $\mathbb{A}$ be a dual Banach algebra, and $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$. Then we call $\Gamma \in \mathcal{L}^2_{w^*}(\mathbb{A}, \mathbb{C})^*$ a ϕ-normal virtual diagonal for $\mathbb{A}$ if $\langle \Omega^{**}_{\mathbb{A}}(\Gamma), \phi \rangle = 1$ and $\mathbf{a}.\Gamma = \phi(\mathbf{a}).\Gamma$ for every $\mathbf{a} \in \mathbb{A}$ [5, Definition 2.4].

THEOREM 3.5. *Suppose that* $\mathbb{A} = (\mathbb{A}_*)^*$ *and* $\mathbb{B} = (\mathbb{B}_*)^*$ *are dual Banach algebras,* $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ *and* $\psi \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$*. Then* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *is* $\phi \otimes \psi$*-strongly Connes amenable if* $\mathbb{A}$ *has a* ϕ*-normal virtual diagonal and* $\mathbb{B}$ *has a* ψ*-normal virtual diagonal and vice versa.*

The next result shows that for an unital dual Banach algebra $\mathbb{A}$, there exists the relation between two defined diagonals in this note, i.e, ϕ-normal virtual diagonal and ϕ-σwc virtual diagonal where, $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$.

PROPOSITION 3.6. *Suppose that* $\mathbb{A}$ *is an arbitrary unital dual Banach algebra, and* $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$*.* $\mathbb{A}$ *has a* ϕ*-normal virtual diagonal if and only if* $\mathbb{A}$ *has a* ϕ*-*σwc *virtual diagonal.*

PROOF. Let $\mathbb{A}$ has a ϕ-σwc virtual diagonal. Then by combining Lemma 2.5 and [5, Theorem 2.7], there exists a ϕ-normal virtual diagonal for $\mathbb{A}$. □

COROLLARY 3.7. *Suppose that* $\mathbb{A}$ *and* $\mathbb{B}$ *are unital dual Banach algebras,* $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ *and* $\psi \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$*. Then* $\mathbb{A}\widehat{\otimes}\mathbb{B}$ *has a* $\phi \otimes \psi$*-normal virtual diagonal if and only if* $\mathbb{A}$ *has a* ϕ*-normal virtual diagonal and* $\mathbb{B}$ *has a* ψ*-normal virtual diagonal.*

3.2. Characterization of θ-Lau product $\mathbb{A} \times_\theta \mathbb{B}$ and (ϕ, θ)-strongly Connes amenability. In this subsection, we study (ϕ, θ)-strongly Connes amenability of Banach algebra $\mathbb{A} \times_\theta \mathbb{B}$, where $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ and $\theta \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$. For this purpose first, we explain the following preliminary notations.

For Banach algebras $\mathbb{A}$ and $\mathbb{B}$ the θ-Lau product $\mathbb{A} \times_\theta \mathbb{B}$ is defined with

$$(\mathbf{a}, \mathbf{b}).(\mathbf{a}', \mathbf{b}') = (\mathbf{a}.\mathbf{a}' + \mathbf{a}.\theta(\mathbf{b}') + \theta(\mathbf{b}).\mathbf{a}', \mathbf{b}\mathbf{b}') \tag{2}$$

and the norm

$$\|(\mathbf{a}, \mathbf{b})\|_{\mathbb{A} \times_\theta \mathbb{B}} = \|\mathbf{a}\|_{\mathbb{A}} + \|\mathbf{b}\|_{\mathbb{B}}, \qquad (\mathbf{a}, \mathbf{a}' \in \mathbb{A}, \mathbf{b}, \mathbf{b}' \in \mathbb{B}).$$

This definition is a certain case of the product that is presented in [8]. Since $\theta \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$, then $\mathbb{A} \times_\theta \mathbb{B}$ is a dual Banach algebra with predual $\mathbb{A}_* \times \mathbb{B}_*$. It is known that $(\mathbb{A} \times_\theta \mathbb{B})^*$ identified with $\mathbb{A}^* \times \mathbb{B}^*$ that

$$\langle (\mathbf{f}, \mathbf{g}), (\mathbf{a}, \mathbf{b}) \rangle = \mathbf{f}(\mathbf{a}) + \mathbf{g}(\mathbf{b}) \tag{3}$$

for all $\mathbf{a} \in \mathbb{A}$, $\mathbf{b} \in \mathbb{B}$ and $\mathbf{f} \in \mathbb{A}^*$, $\mathbf{g} \in \mathbb{B}^*$.

DEFINITION 3.8. Suppose that $\mathbb{A} = (\mathbb{A}_*)^*$ is a dual Banach algebra, and $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$. A linear functional μ on $\mathbb{A}^*$ is called a mean if $\mu(\phi) = 1$. Also, μ is called ϕ-invariant mean if $\mu(f.\mathbf{a}) = \phi(\mathbf{a})\mu(f)$ for all $\mathbf{a} \in \mathbb{A}$ and $f \in \mathbb{A}_*$ [5].

THEOREM 3.9. *Suppose that* $\mathbb{A}$ *and* $\mathbb{B}$ *are dual Banach algebras with predual* $\mathbb{A}_*$ *and* $\mathbb{B}_*$*, respectively. Suppose that* $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$*, and* $\theta \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$*. Then* $\mathbb{A} \times_\theta \mathbb{B}$ *has* (ϕ, θ)*-*σwc *virtual diagonal if and only if* $\mathbb{A}$ *has a* ϕ*-*σwc *virtual diagonal.*

THEOREM 3.10. *Suppose that* $\mathbb{A} = (\mathbb{A}_*)^*$*, and* $\mathbb{B} = (\mathbb{B}_*)^*$ *are dual Banach algebras. Suppose that* $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ *and* $\theta \in \Delta_{w^*}(\mathbb{B}) \cap \mathbb{B}_*$*, then the following two conditions are hold:*

(a) *if* $\mathbb{A} \otimes_\theta \mathbb{B}$ *is* $(0, \theta)$*-strongly Connes amenable, then* $\mathbb{B}$ *is* θ*-strongly Connes amenable;*

(b) *if* $\mathbb{A}$ *is unital and* $\mathbb{B}$ *is* θ*-strongly Connes amenable, then* $\mathbb{A} \otimes_\theta \mathbb{B}$ *is* $(0, \theta)$*-strongly Connes amenable.*

4. Characterization of Banach algebras of module extention

In this section we investigate ϕ-strongly Connes amenability for dual Banach algebras of module extension. Let $\mathbb{A} = (\mathbb{A}_*)^*$ be a dual Banach algebra, and $\mathbb{X} = (\mathbb{X}_*)^*$ be a normal Banach $\mathbb{A}$-bimodule. We denote the l^1-direct sum of a Banach algebra $\mathbb{A}$ with a nonzero Banach $\mathbb{A}$-bimodule $\mathbb{X}$ by $\mathbb{A} \oplus \mathbb{X}$. We define the algebraic product and norm for $\mathbb{A} \oplus \mathbb{X}$ as follows:

$$(\mathbf{a}, \mathbf{x}) \cdot (\mathbf{a}', \mathbf{x}') = (\mathbf{a}\mathbf{a}', \mathbf{a}\mathbf{x}' + \mathbf{x}\mathbf{a}'), \qquad (\mathbf{a}, \mathbf{a}' \in \mathbb{A}, \mathbf{x}, \mathbf{x}' \in \mathbb{X})$$

and

$$\|(\mathbf{a}, \mathbf{x})\| = \|\mathbf{a}\|_{\mathbb{A}} + \|\mathbf{x}\|_{\mathbb{X}}.$$

With above structure, $\mathbb{A} \oplus \mathbb{X}$ is called a Banach algebra of module extension. Some algebras of this form have been discussed in [1] and [10]. It is known that $\mathbb{A} \oplus \mathbb{X} = (\mathbb{A}_* \oplus_\infty \mathbb{X}_*)^*$ is a dual Banach algebra, where $\oplus_\infty$ denote l_∞-direct sum of Banach $\mathbb{A}$-bimodules. Dual and the second dual of $\mathbb{A} \oplus \mathbb{X}$ identified with $(\mathbb{A} \oplus \mathbb{X})^* = \mathbb{A}^* \oplus_\infty \mathbb{X}^*$ and $(\mathbb{A} \oplus \mathbb{X})^{**} = \mathbb{A}^{**} \oplus \mathbb{X}^{**}$, respectively. It follows that $\mathbb{X}^{**}$ is Banach $\mathbb{A}^{**}$-bimodule, where $\mathbb{A}^{**}$ is equipped with the first Arens product(for more details see [10]).

In [3], the authors showed that $\Delta(\mathbb{A} \oplus \mathbb{X}) = \Delta(\mathbb{A}) \times \{0\}$ and they proved that if $\mathbb{A} \oplus \mathbb{X}$ is $(\phi, 0)$-amenable then $\mathbb{A}$ is ϕ-amenable where $\phi \in \Delta_{w^*}(\mathbb{A}) \cap \mathbb{A}_*$ and the converse also holds if $\mathbb{X}\mathbb{A} = 0$. One might ask whether the result extends to $(\phi, 0)$-module Connes amenability. We give an affirmative answer to this question.

Our purpose from the next theorem is to investigate the relation between the $(\phi, 0)$-σwc virtual diagonal of Banach algebra of module extension $\mathbb{A} \oplus \mathbb{X}$ and the ϕ-σwc virtual diagonal of Banach algebra $\mathbb{A}$.

Theorem 4.1. *Suppose that $\mathbb{A} = (\mathbb{A}_*)^*$ is a dual Banach algebra, and $\phi \in \Delta_{\omega^*}(\mathbb{A}) \cap \mathbb{A}_*$. Suppose that $\mathbb{X}$ is a normal Banach $\mathbb{A}$-bimodule with predual $\mathbb{X}_*$. Then,*

(a) *if there exists a $(\phi, 0)$-σwc virtual diagonal for $\mathbb{A} \oplus \mathbb{X}$, then there exists a ϕ-σwc virtual diagonal for $\mathbb{A}$;*

(b) *if $\mathbb{X}\mathbb{A} = 0$ and $\mathbb{A}$ has a ϕ-σwc virtual diagonal, then $\mathbb{A} \oplus \mathbb{X}$ has a $(\phi, 0)$-σwc virtual diagonal.*

Acknowledgement

The authors would like to thank the referee for his/her careful reading of the paper.

References

1. W. G. Bade, P. C. Curtis, JR and H. G. Dales, *Amenability and weak amenability for Beurling and Lipshitz algebras*, Proceedings of the London Mathematical Society, **s3-55**(2) (1987), 359-–377.
2. G. Corach and J. E. Gale, *Averaging with virtual diagonals and geometry of representations*, in: E. Albrecht and M. Mathieu (eds.), *Banach Algebras* 97, Walter de Grutyer, Berlin, 1998.
3. H. R. Ebrahimi Veshki and A. R. Khoddami, *Character inner amenability of certain Banach algebras*, Colloquium Mathematicum, **122** (2011), 225–232.
4. E. G. Effros, *Amenability and virtual diagonals for von Neumann algebras*, J. Funct. Anal. **78**(1) (1988), 137–53.
5. A. Ghaffari and S. Javadi,*ϕ-Connes amenability of dual Banach algebras*, Bulletin of the Iranian Mathematical Society, **43**(1) (2017), 25–39.
6. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., **127** (1972).
7. A. Mahmoodi, *On ϕ-Connes amenability of dual Banach algebras*, Journal of Linear and Topological Algebra, **4** (2014), 211–217.
8. M. S. Monfared, *On certain products of Banach algebras with applications to harmonic analysis*, Studia Math., **178** (2007), 277–294.
9. V. Runde, *Connes amenability and normal, virtual diagonals for measure algebras I*, J. London Math. Soc., **67**(1) (2003), 643-–656.
10. Y. Zhang, *Weak amenability of module extensions of Banach algebras*, Trans. Amer. Math. Soc., **354** (2002), 4131–4151.

E-mail: `e.tamimi@velayat.ac.ir`