χ-Connes Module Amenability of Semigroup Algebras

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Abstract. In this paper, we define χ-Connes module amenability of a semigroup algebra $l^1(S)$, where χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is ω^*-continuous and S is an inverse weakly cancellative semigroup with subsemigroup E of idempotents. We are mainly concerned with the study of χ-module normal, virtual diagonals. We show that if $l^1(S)$ as a Banach module over $l^1(E)$ is χ-Connes module amenable, then it has a χ-module normal, virtual diagonal. In the case $\chi = id$, the converse also holds.

Keywords: χ-Connes module amenable, χ-module normal virtual diagonal, Inverse semigroup algebra, Module φ-derivation, Weakly cancellative semigroup.
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1. Introduction

In [1], Amini introduced the concept of module amenability for Banach algebras, and proved that when S is an inverse semigroup with subsemigroup E of idempotents, then $l^1(S)$ as a Banach module over $U = l^1(E)$ is module amenable if and only if S is amenable. We may refer the reader e.g. to [1, 2, 6], for extensive treatments of various notions of module amenability. All of these concepts generalized the earlier concept of amenability for Banach algebras introduced by Johnson [4]. In this paper, we introduce the concept of χ-Connes module amenability for semigroup algebra $l^1(S)$ and give a characterization of χ-Connes module amenability in terms of χ-modul normal virtual diagonals. In particular, we show that if χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is ω^*-continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ-Connes module amenable, then it has a χ-module normal virtual diagonal. In the case $\chi = id$, the converse also holds.

2. Main results

Let S be a semigroup. Then S is named cancellative semigroup, if for every $r, s \neq t \in S$ we have $rs \neq rt$ and $sr \neq tr$.

A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotent elements of S is denoted by E. For $s \in S$, we define $L_s, R_s : S \to S$ by $L_s(t) = st, R_s(t) = ts; (t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative.

Before turning our result, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$[3].

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let I be the closed ideal of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ generated by elements of the form $\alpha.(a \otimes b) - (a \otimes b).\alpha$, for $a, b \in \mathcal{A}$ and $\alpha \in \mathcal{U}$. $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{A}$ is defined to be the quitient Banach space $\frac{\mathcal{A}\widehat{\otimes}\mathcal{A}}{I}$.

Let J be the closed ideal of $\mathcal{A}$ generated by elements of the form $(\alpha.a).b - a.(b.\alpha)$. Since J is ω^*-closed, then the quotient algebra $\frac{\mathcal{A}}{J}$ is again dual with predual ${}^{\perp}J = \{\phi \in \mathcal{A}_* : \langle \phi, a\rangle = 0 \ for\ all\ \ a \in J\}$. Also we have $J^{\perp} = \{\phi^* \in \mathcal{A}^* : \langle \phi, a\rangle = 0 \ for\ all\ \ \phi \in J\}$.

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Definition 2.1. Let $\mathcal{A}$ be a dual Banach algebra. A module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is a map $\varphi : \mathcal{A} \to \mathcal{A}$ with

$$\varphi(\alpha.a + b.\beta) = \alpha.\varphi(a) + \varphi(b).\beta, \quad \varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

In this paper we let that $\mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})$ denote the separately ω^*-continuous two-linear maps from $\frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J}$ to $\mathbb{C}$, $\tilde{\omega} : \mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{A} \to \frac{\mathcal{A}}{J}$ be the multiplication operator with $\tilde{\omega}(a \otimes b + I) = ab + J$ and $\tilde{\varphi} : \frac{\mathcal{A}}{J} \to \frac{\mathcal{A}}{J}$ be the map that is defined by $\tilde{\varphi}(a + J) = \varphi(a) + J, \ \ a \in \mathcal{A}$.

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra and $\varphi : \mathcal{A} \to \mathcal{A}$ be a bounded ω^*-continuous module homomorphism. An element $M \in \mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})^*$ is called a φ-module normal virtual diagonal for $\mathcal{A}$ if $\tilde{\omega}^{**}(M)$ is an identity for $\frac{\varphi(\mathcal{A})}{J}$ and

$$M.\tilde{\varphi}(c + J) = \tilde{\varphi}(c + J).M \qquad (c \in \mathcal{A}).$$

Let X be a dual Banach $\mathcal{A}$-bimodule. X is called normal if for each $x \in X$, the maps

$$\mathcal{A} \to X; \qquad a \to a.x, \quad a \to x.a$$

are ω^*-continuous. If moreover X is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in X$

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

then X is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly for the right and two sided actions. Also, X is called symmetric, if $\alpha.x = x.\alpha \qquad (\alpha \in \mathcal{U}, x \in X)$.

Throughout this paper $\mathcal{H}_{\omega^*}(\mathcal{A})$ will denotes the space of all bounded module homomorphisms from $\mathcal{A}$ to $\mathcal{A}$ that are ω^*-continuous.

Definition 2.3. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\varphi \in \mathcal{H}_{\omega^*}(\mathcal{A})$ and let that X be a dual Banach $\mathcal{A}$-bimodule. A bounded map $D_{\mathcal{U}} : \mathcal{A} \to X$ is called a module φ-derivation if for every $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathcal{U}$, we have

$$D_{\mathcal{U}}(\alpha.a \pm b.\beta) = \alpha.D_{\mathcal{U}}(a) \pm D_{\mathcal{U}}(b).\beta, \quad D_{\mathcal{U}}(ab) = D_{\mathcal{U}}(a).\varphi(b) + \varphi(a).D_{\mathcal{U}}(b).$$

When X is symmetric, each $x \in X$ defines a module φ-derivation

$$(D_{\mathcal{U}})_x(a) = \varphi(a).x - x.\varphi(a) \qquad (a \in \mathcal{A}).$$

Derivations of this form are called inner module φ-derivation.

Definition 2.4. Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module and $\varphi \in \mathcal{H}_{\omega^*}(\mathcal{A})$. $\mathcal{A}$ is called φ-Connes module amenable if for any symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module X, each ω^*-continuous module φ-derivation $D_{\mathcal{U}} : \mathcal{A} \to X$ is inner.

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}$ be a unital dual Banach $\mathcal{U}$-module and let $\mathcal{A}$ has an id-module normal virtual diagonal. Then $\mathcal{A}$ is id-Connes module amenable.*

Proof. Let X be a symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module. We first note that $\mathcal{A}$ has an identity. It is therefore sufficient for $\mathcal{A}$ to be id-Connes module amenable that we suppose that X is unital. Let $D_{\mathcal{U}} : \mathcal{A} \to X$ be a module derivation that is ω^*-continuous. It is straightforward to see that E is a normal Banach $\frac{\mathcal{A}}{J}$-$\mathcal{U}$-module. Let $X = (X_*)^*$. Since X is symmetric, then $D_{\mathcal{U}}|_J = 0$. We define $\tilde{D_{\mathcal{U}}} : \frac{\mathcal{A}}{J} \to X; \ \tilde{D_{\mathcal{U}}}(a + J) := D_{\mathcal{U}}(a) \ (a \in \mathcal{A})$. To each $x \in X_*$, there corresponds $V_x : \frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J} \to \mathbb{C}$ via $V_x(a + J, b + J) = \langle x, (a + J)\tilde{D_{\mathcal{U}}}(b + J)\rangle (a, b \in \mathcal{A})$. It is clearly that $V_x \in \mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})$. For each $a, b \in \mathcal{A}$ and $a_* \in \mathcal{A}_*$ we have

$$\langle \int ab + JdM, a_* + J^{\perp}\rangle = \langle M, \tilde{\omega}^*(a_* + J^{\perp})\rangle = \langle \tilde{\omega}^{**}(M), a_* + J^{\perp}\rangle.$$

Now, put $f(x) = \langle M, \nu_x\rangle (x \in X_*)$. Let $c \in \mathcal{A}$. After a little calculation, we obtain

$$\langle (c + J).f - f.(c + J)\rangle = \int \langle (ab + J)\tilde{D_{\mathcal{U}}}(c + J), x\rangle dM = \langle \tilde{\omega}^{**}(M).\tilde{D_{\mathcal{U}}}(c + J), x\rangle.$$

All in all, $D_{\mathcal{U}}(c) = c.f - f.c$ holds. □

In Theorem 2.5 it is shown that if a unital Banach algebra $\mathcal{A}$ has an id-module normal virtual diagonal, then $\mathcal{A}$ is id-Connes module amenable. Let S be a semigroup, it would be interesting to know that the converse holds for inverse semigroup algebra $l^1(S)$. Thus for an inverse semigroup S, we consider an equivalence relation on S where $s \sim t$ if and only if there is $e \in E$ such that $se = te$. The quotient semigroup $S_G = \frac{S}{\sim}$ is a group [5]. Also, E is a symmetric subsemigroup of S. Therefore, $l^1(S)$ is a Banach $l^1(E)$-module with compatible canonical actions. Let $l^1(E)$ acts on $l^1(S)$ via

$$\delta_e.\delta_s = \delta_s, \qquad \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e \qquad\qquad (s \in S, e \in E).$$

With above notation, $l^1(S_G)$ is a quotient of $l^1(S)$ and so the above action of $l^1(E)$ on $l^1(S)$ lifts to an action of $l^1(E)$ on $l^1(S_G)$, making it a Banach $l^1(E)$-module [1].

The following theorem is the main result of the present paper.

THEOREM 2.6. *Let S be a weakly cancellative semigroup. Let S be an inverse semigroup with idempotents E, let $l^1(S)$ be a Banach $l^1(E)$- module and let $\chi \in \mathcal{H}_{\omega^*}(l^1(S))$. If $l^1(S)$ is χ-Connes module amenable, then $l^1(S)$ has a χ-module normal virtual diagonal.*

PROOF. Let $\pi : S \to S_G$ be the quotient map. By [1, Lemma 3.2], we define a bimodule action of $l^1(S)$ on $l^\infty(S_G)$ by

$$\delta_s.x = \delta_{\pi(s)} * x, \qquad x.\delta_s = x * \delta_{\pi(s)} \qquad\qquad (s \in S, x \in l^\infty(S_G)).$$

Since $c_0(S_G)$ is an introverted subspace of $l^1(S_G)$ then $l^1(S_G)^*$ is a normal Banach $l^1(S)$-$l^1(E)$-module. Choose $n \in l^1(S_G)^*$ such that $\langle n, 1\rangle = 1$, and define $D : l^1(S) \to l^1(S_G)^*$ by $D(\delta_s) = \chi(\delta_s).n - n.\chi(\delta_s)$. Moreover, D attains its values in the ω^*-closed submodule $(\frac{l^\infty(S_G)}{\mathbb{C}})^*$. Since $l^1(S)$ is χ-Connes module amenable, then D is inner. Consequently, there exists $\tilde{n} \in (\frac{l^\infty(S_G)}{\mathbb{C}})^*$ such that $D(\delta_s) = ad_{\tilde{n}}$, so

$$\tilde{\chi}(\delta_{\pi(s)}).n - n.\tilde{\chi}(\delta_{\pi(s)}) = \tilde{\chi}(\delta_{\pi(s)}).\tilde{n} - \tilde{n}.\tilde{\chi}(\delta_{\pi(s)})$$

Then we may define

$$\langle M, f\rangle = \lim_\alpha \int f\big(\tilde{\chi}(\delta_{\pi(x^*)}), \tilde{\chi}(\delta_{\pi(x)})\big) f_\alpha(x)dx.$$

Also for each s we obtain

$$\begin{aligned}\tilde{\omega}^{**}(M).\tilde{\chi}(\delta_{\pi(s)}) = \langle M, \tilde{\omega}^*(\tilde{\chi}(\delta_{\pi(s)}))\rangle &= \lim_\alpha \int (\omega^*(\tilde{\chi}(\delta_{\pi(s)})))(\tilde{\chi}(\delta_{\pi(x^*)})), \tilde{\chi}(\delta_{\pi(x)}))f_\alpha(x)dx \\ &= \lim_\alpha \tilde{\chi}(\delta_{\pi(s)}) \int f_\alpha(x)dx = \tilde{\chi}(\delta_{\pi(x)}).\end{aligned}$$

Consequently, M is a χ-normal module virtual diagonal for $l^1(S)$. □

THEOREM 2.7. *Let S be a weakly cancellative semigroup with idempotents E, let $l^1(S)$ be a unital dual Banach $l^1(E)$-module and let $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ be a dual Banach $l^1(E)$-module and $\chi \in \mathcal{H}_{\omega^*}(l^1(S))$. If $l^1(S)$ is χ-Connes module amenable, then $l^1(S)\widehat{\otimes}_{l^1(E)}l^1(S)$ is $\chi\otimes_{l^1(E)}\chi$-Connes module amenable.*

COROLLARY 2.8. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E and let $l^1(S)$ be a Banach $l^1(E)$-module. Then $l^1(S)$ is Connes module amenable if and only if $l^1(S)$ has a module normal virtual diagonal.*

PROOF. This follows immediately from Theorem 2.5 and Theorem 2.6. □

EXAMPLE 2.9. Let $(\mathbb{N}, \vee)$ be the semigroup of positive integers with maximum operation. Since $\mathbb{N}$ is weakly cancellative, then $l^1(\mathbb{N})$ is a dual Banach algebra with predual $c_0(\mathbb{N})$. By [3, Theorem 5.13], $l^1(\mathbb{N})$ is not Connes amenable. Moreover $l^1(\mathbb{N})$ is module amenable on $l^1(E_\mathbb{N})$, so it is Connes module amenable.

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