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On l^1-Munn Algebras and Connes Amenability

Ebrahim Tamimi *
Faculty of Mathematics and Computer Science, Velayat University, Iranshahr, Iran

Abstract. In this paper, we study and investigate Connes amenability for l^1-Munn algebra $\mathcal{LM}(\mathcal{A}; P, I, J)$ where $\mathcal{A}$ is a Banach algebra, I and J are nonempty sets and P is invertible matrix. Then we use the obtained results to semigroup algebras $l^1(S)$ that S is a semigroup. Also, we prove that if S is a weakly cancellative semigroup and $l^1(S)$ is Connes amenable then the idempotents set of S will be finite.
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1. Introduction

l^1-Munn algebras are introduced by Eslamzadeh in [2]. Eslamzadeh characterized amenable semigroup algebras by l^1-Munn algebras. In [5], Munn introduced a certain type of mentioned algebras. l^1-Munn algebras has been investigated in some texts. For instance, Blackmore showed then l^1-Munn algebra of the group algebra $l^1(G)$ is weakly amenable. Also, the structure of this algebras studied by Eslamzadeh in [3]. On the other hand, the authors in [1] used the l^1-Munn algebras to study of semigroup algebras of completely simple semigroups. We know that special concept of amenability was called Connes amenability. In [6], Runde extended the notion of Connes-amenability to dual Banach algebras. For a locally compact group G, the measure algebra $M(G)$ is an example of a dual Banach algebra. Runde introduced normal, virtual diagonals for a dual Banach algebra and showed that the existence of a normal virtual diagonal for $M(G)$ is equivalent to it being Connes amenable. In particular, $l^1(G)$ is amenable if and only if $l^1(G)$ is Connes amenable. In this paper we investigate the semigroup algebra of a weakly cancellative semigroup in theorem of Runde. Also, the concept of Connes amenability of l^1-Munn algebras is investigated. We apply the l^1-Munn algebras to study of Connes amenability of mentioned semigroup algebras.

2. Main results

Let $\mathcal{A}$ be a dual Banach algebra such that $(\mathcal{A}_*)^* = \mathcal{A}$. A dual Banach $\mathcal{A}$-bimodule X is called normal Banach $\mathcal{A}$-bimodule if for each $x \in X$, the maps $a \to a.x,\ \ a \to x.a$ are ω^*-continuous $(a \in \mathcal{A})$.

Now, in the following we present some definitions.

Definition 2.1. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and X be a Banach $\mathcal{A}$-bimodule. The linear bounded map $D : \mathcal{A} \to X$ is called derivation if for every $a, b \in \mathcal{A}$ we have $D(ab) = D(a).b + a.D(b)$. We say D is inner derivation if there exists $x \in X$ such that for every $a \in \mathcal{A}$, $D(a) = a.x - x.a$. In this case we denote D by ad_x.

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. $\mathcal{A}$ is called amenable if every derivation $D : \mathcal{A} \to X^*$ is inner.

Definition 2.2. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and X be a normal dual Banach $\mathcal{A}$-bimodule. $\mathcal{A}$ is called Connes amenable if every ω^*-continuous derivation $D : \mathcal{A} \to X$ is inner.

*speaker

Remark 2.3. Let $\mathcal{A}$ be a dual Banach algebra and X be a Banach $\mathcal{A}$-bimodule. An element $x \in X$ is called ω^*-weakly continuous if the module maps $a \to x.a, a \to a.x$ are ω^*-weak continuous $(a \in \mathcal{A})$. The collection of all ω^*-weakly continuous elements of X is denoted by $\sigma wc(X)$. We define the map $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$ by $\pi(a \otimes b) = ab$, $(a, b \in \mathcal{A})$. Then we have $\pi^* : \mathcal{A}_* \to \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$. Consequently, π^{**} drops to a homomorphism $\pi_{\sigma wc} : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^* \to \mathcal{A}$.

Definition 2.4. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A}^*)^*)$ is called a σwc-virtual diagonal for $\mathcal{A}$, if $M.u = u.M$ for every $u \in \mathcal{A}$ and $u.\pi_{\sigma wc}(M) = u$ for every $u \in \mathcal{A}$.

It is shown that Connes amenability of Banach algebra $\mathcal{A}$ is equal to existence a σwc-virtual diagonal for $\mathcal{A}$.

Definition 2.5. Let $\mathcal{A}$ be a unital Banach algebra, let $I \neq \varnothing$, $J \neq \varnothing$ be arbitrary sets and $P = (p_{ij}) \in M_{J\times I}(\mathcal{A})$ be a matrix such that $\| P \|_\infty = \sup\{\| p_{ji} \| : j \in J, i \in I\} \leq 1$. The set $M_{I\times J}(\mathcal{A})$ of all $I \times J$ matrices $a = (a_{ij})$ on $\mathcal{A}$ with l^1-norm and the product $A \bullet B = APB$, $(A, B \in M_{I\times J}(\mathcal{A}))$ is a Banach algebra that is called l^1-Munn algebra on $\mathcal{A}$ with sandwich matrix P. In this case l^1-Munn algebra is denoted by $\mathcal{LM}(\mathcal{A}; P, I, J)$ [2].

The following statements are from [2].
A semigroup S is called regular if for every $a \in S$ there is $b \in S$ such that $a = aba$. S is an inverse semigroup if for every $a \in S$ there is a unique $a^* \in S$ such that $aa^*a = a$ and $a^*aa^* = a^*$. Let G be a group, $I \neq \varnothing$ and $J \neq \varnothing$ be arbitrary sets, and $G^0 = G \cup \{0\}$ be the group with zero arising from G by adjunction of a zero element. An $I \times J$ matrix A over G^0 that has at most one nonzero entry $a = A(i, j)$ is called a Rees $I \times J$ matrix over G^0 and is denoted by $(a)_{ij}$. Let P be a $J \times I$ matrix over G^0. The set $S = G \times I \times J$ with the composition $(a, i, j) \circ (b, l, k) = (aP_{jl}b, i, k)$, $(a, i, j), (b, l, k) \in S$ is a semigroup that we denote by $\mathcal{M}(G, P)$ [4, p. 68]. Similarly if P is a $J \times I$ matrix over G^0, then $S = G \times I \times J \cup \{0\}$ is a semigroup under the following composition operation:

$$(a, i, j) \circ (b, l, k) = \begin{cases} (aP_{jl}b, i, k); & P_{jl} \neq 0 \\ 0; & P_{jl} = 0 \end{cases}$$

$$(a, i, j) \circ 0 = 0 \circ (a, i, j) = 0 \circ 0 = 0$$

Mentioned semigroup which is denoted by $\mathcal{M}^0(G, P)$ also can be described in the following way: The set of all Rees $I \times J$ matrices over G^0 form a semigroup under the binary operation $A \bullet B = APB$, which is called the Rees $I \times J$ matrix semigroup over G^0 with the sandwich matrix P and is isomorphic to $\mathcal{M}^0(G, P)$ [4, pp. 61-63]. An $I \times J$ matrix P over G^0 is called regular (invertible) if every row and every column of P contains at least (exactly) one nonzero entry.

Already in[2], Esslamzadeh shows that if $\mathcal{LM}(\mathcal{A}; P, I, J)$ is amenable then $\mathcal{A}$ is amenable, $I \neq \varnothing$ and $J \neq \varnothing$ are finite and P is invertible. In the following theorem we investigate this subject for Connes amenablility of a Banach algebra.

Theorem 2.6. *Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{LM}(\mathcal{A}; P, I, J)$ be Connes amenable. Then $\mathcal{A}$ is Connes amenable, moreover the sets I and J are finite and matrix P is invertible.*

Proof. Let $\mathcal{LM}(\mathcal{A}; P, I, J)$ be Connes amenable. Thus it has a bounded approximate identity. By applying [2, Lemma 3.7], we implise that I and J are finite sets. So, P must be invertible. For prove Connes amenability of $\mathcal{A}$, it is sufficient that suppose $X = (X_*)^*$ be a normal Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to X$ be a ω^*-continuous derivation. We show that D is inner. □

If in the above theorem, $\mathcal{A}$ has an identity element then we conclude the following corollary.

Corollary 2.7. *Let $\mathcal{A}$ be a Connes amenable unital dual Banach algebra, I and J are finite sets and P is invertible matrix. Then Banach algebra $\mathcal{LM}(\mathcal{A}; P, I, J)$ is Connes amenable.*

Proof. Without losing the generality let $|I| = |J| = n$. Suppose that X be a normal Banach $\mathcal{LM}(\mathcal{A}; I_n, I, J)\widehat{\otimes}\mathcal{A}$-bimodule and $D : \mathcal{LM}(\mathcal{A}; I_n, I, J)\widehat{\otimes}\mathcal{A} \to X$ be a ω^*-continuous derivation. Now, we define the right and left module actions on $\mathcal{A}$, then we prove that D is inner. This complete the proof. □

Let S be a semigroup. We say that S is cancellative semigroup, if for every $r, p \neq q \in S$ we have $rs \neq rq$ and $pr \neq qr$.

An element $e \in S$ is called an idempotent if $e = e^* = e^2$. For $p \in S$, we define $L_p, R_p : S \to S$ by $L_p(q) = pq, R_p(q) = qp; (q \in S)$. If for each $p \in S$, L_p and R_p are finite-to-one maps, in this case S is named weakly cancellative. Suppose that S be a semigroup. S is called simple semigroup if only ideal in S is itself. S is called completely simple semigroup if it is simple and contain a principle idempotent.

LEMMA 2.8. *Let S be a weakly cancellative semigroup and let $l^1(S)$ be Connes amenable unital semigroup algebra. Then the idempotents set S is finite.*

THEOREM 2.9. *Let G be a group and S be a weakly cancellative semigroup and let $l^1(S)$ be Connes amenable unital semigroup algebra. Then S is a Rees matrix semigroup of the form $S = \mathcal{M}(G; P, I, J)$ and l^1-Munn algebra $\mathcal{LM}(l^1(G); P, I, J)$ has an identity.*

PROOF. By above lemma, S is a simple semigroup with finite idempotents set, thus S is a completely simple semigroup. Therefore, S is a Rees matrix semigroup of the form $S = \mathcal{M}(G; P, I, J)$. This complete the proof. □

THEOREM 2.10. *Let S be a weakly cancellative semigroup with finite idempotents set. Let $l^1(S)$ be an unital semigroup algebra. Moreover, suppose that S be a Rees matrix semigroup of the form $S = \mathcal{M}(G; P, I, J)$. If $l^1(S)$ is Connes amenable then $l^1(S)$ is amenable and vice versa.*

PROOF. If $l^1(S)$ is Connes amenable then by Theorem 2.6 and [2] the proof is clear. The converse follows directly from the above theorem. □

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E-mail: e.tamimi@velayat.ac.ir