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On module Connes amenability of Banach algebras of module extension and Θ-Lau product of dual Banach algebras

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ABSTRACT. In this paper, we develop the notions of module Connes amenability for certain Banach algebras. We investigate the module Connes amenability of Θ-Lau product $\mathcal{A} \times_\Theta \mathcal{B}$ and Banach algebra of module extension $\mathcal{A} \oplus \mathcal{X}$, where $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras with preduals $\mathcal{A}_*$ and $\mathcal{B}_*$ also, $\Theta : \mathcal{B} \to \mathcal{A}$ is an algebraic homomorphism and $\mathcal{X}$ is a normal Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$.

Keywords: module Connes amenability, module extension Banach algebra, φ-derivation, Θ-Lau product

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1. Introduction

The concept of Connes amenability of dual Banach algebras is introduced by V. Runde and have since then turned out to be very interesting objects of research. A Banach $\mathcal{A}$-bimodule E is called normal if it is a dual space such that the module actions

$$\mathcal{A} \longrightarrow E; \qquad a \to a.x, \ a \to x.a \qquad (a \in \mathcal{A}, x \in E)$$

are separately $weak^*$-continuous. A dual Banach algebra $\mathcal{A}$ is called Connes amenable if every $weak^*$-continuous derivation from $\mathcal{A}$ into a normal, dual Banach $\mathcal{A}$-bimodule is inner. Let E be a Banach $\mathcal{A}$-bimodule. The collection of all elements of E that for those, the module maps from $\mathcal{A}$ to E are $weak^*$-weakly continuous is denoted by $\sigma wc(E)$.

Recently the authors have introduced the ϕ-version of Connes amenability of dual Banach algebra $\mathcal{A}$ where ϕ is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$ and $\phi \in \mathcal{A}_*$ [4]. A dual Banach algebra $\mathcal{A}$ is ϕ-Connes amenable if, for every normal $\mathcal{A}$-bimodule E where the left action is of the form $a.x = \phi(a)x$, every bounded $weak^*$-continuous derivation $D : \mathcal{A} \to E$ is inner.

In this paper, we are going to investigate the module Connes amenability and character module Connes amenability for $\mathcal{A} \times_\Theta \mathcal{B}$ and $\mathcal{A} \oplus \mathcal{X}$.

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2. Main results

Let $\mathcal{A}$ be a Banach algebra. Let $\mathcal{K}$ be a subspace of $\mathcal{A}^*$ and $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{K}$. A linear functional m on $\mathcal{K}$ is called a mean if $\langle m, \varphi\rangle = 1$. A mean m is φ-invariant mean if $\langle m, a.f\rangle = \varphi(a)\langle m, f\rangle$ for all $a \in \mathcal{A}$ and $f \in \mathcal{K}$. Note that if $\mathcal{K} = \mathcal{A}^*$, then $\mathcal{A}$ is φ-amenable and if $\mathcal{K} = \mathcal{A}_*$, then $\mathcal{A}$ is φ-Connes amenable(see [5] and [4]).

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via the following compatible actions

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let E be a normal dual Banach $\mathcal{A}$-bimodule. Also, if E is an $\mathcal{U}$-bimodule such that

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in E$ then E is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly, for the right and two-sided actions. E is called symmetric, if $\alpha.x = x.\alpha$ $(\alpha \in \mathcal{U}, x \in E)$.

Definition 2.1. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule, $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{K}$ and E be a dual Banach $\mathcal{A}$-$\mathcal{U}$-module. A bounded map $D_{\mathcal{U}} : \mathcal{A} \to E$ is called a module φ-derivation if

$$D_{\mathcal{U}}(\alpha.a \pm b.\beta) = \alpha.D_{\mathcal{U}}(a) \pm D_{\mathcal{U}}(b).\beta, \qquad D_{\mathcal{U}}(ab) = D_{\mathcal{U}}(a).\varphi(b) + \varphi(a).D_{\mathcal{U}}(b),$$

for all $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathcal{U}$.

Suppose that E is symmetric then, the module φ-derivation $D_{\mathcal{U}}$ is called inner if there exists $x \in E$ such that for every $a \in \mathcal{A}$ we have $(D_{\mathcal{U}})_x(a) = \varphi(a).x - x.\varphi(a)$. Derivations of this form are called inner module φ-derivation.

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module and $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{K}$. $\mathcal{A}$ is called φ-module Connes amenable if for any symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module E, every *weak**-continuous module φ-derivation $D_{\mathcal{U}} : \mathcal{A} \to E$ is inner.

2.1. module Connes amenability of Θ-Lau product $\mathcal{A} \times_\Theta \mathcal{B}$.

Let $\mathcal{A}$ be an unital dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{B}$ be a dual Banach algebra with predual $\mathcal{B}_*$. Suppose that $\theta \in \mathcal{B}_* \cap \Delta(\mathcal{B})$ and consider algebraic homomorphism $\Theta : \mathcal{B} \to \mathcal{A}$ defined by $\Theta(b) = \theta(b)e_{\mathcal{A}}$. We define Θ-Lau product $\mathcal{A} \times_\Theta \mathcal{B}$ by

$$(a_1, b_1).(a_2, b_2) = (a_1.a_2 + a_1.\Theta(b_2) + \Theta(b_1).a_2, b_1b_2)$$

and the norm $\|(a_1, b_1)\|_{\mathcal{A}\times_\Theta\mathcal{B}} = \|a_1\|_{\mathcal{A}} + \|b_1\|_{\mathcal{B}}$ for all $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$. Since $\theta \in \mathcal{B}_* \cap \Delta(\mathcal{B})$, then $\mathcal{A} \times_\Theta \mathcal{B}$ is a dual Banach algebra with predual $\mathcal{A}_* \times \mathcal{B}_*$. It is known that $(\mathcal{A} \times_\Theta \mathcal{B})^* \approx \mathcal{A}^* \times \mathcal{B}^*$ that $\langle (f, g), (a, b)\rangle = f(a) + g(b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$ and $f \in \mathcal{A}^*, g \in \mathcal{B}^*$. Also, $(\mathcal{A} \times_\Theta \mathcal{B})^{**} \approx \mathcal{A}^{**} \times_{\Theta^{**}} \mathcal{B}^{**}$. Consider $\mathcal{A} \approx \mathcal{A} \times \{0\}$, then $\mathcal{A}$ is a closed ideal in $\mathcal{A} \times_\Theta \mathcal{B}$ and $(\mathcal{A} \times_\Theta \mathcal{B})/\mathcal{A}$ is isometric isomorphism with $\mathcal{B}$. In the following we study module Connes amenability of $\mathcal{A} \times_\Theta \mathcal{B}$.

Theorem 2.3. *Let $\mathcal{A}$ be an unital dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{B}$ be a dual Banach algebra with predual $\mathcal{B}_*$. Then the following two statements are equivalent.*

(i) *$\mathcal{A}$ and $\mathcal{B}$ are module Connes amenable.*
(ii) *$\mathcal{A} \times_\Theta \mathcal{B}$ is module Connes amenable.*

It is shown that $\Delta(\mathcal{A} \times_\Theta \mathcal{B}) = \{(\psi, \theta) : \psi \in \Delta(\mathcal{A})\} \cup \{(0, \varphi) : \varphi \in \Delta(\mathcal{B})\}$. We can prove the next result.

Corollary 2.4. *Let $\mathcal{A}$ be a unital dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{B}$ be a dual Banach algebra with predual $\mathcal{B}_*$. Let $\psi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$ and $\theta \in \Delta(\mathcal{B}) \cap \mathcal{B}_*$. Then*

(i) *$\mathcal{A} \times_\Theta \mathcal{B}$ is (ψ, θ)-module Connes amenable if and only if $\mathcal{A}$ is ψ-module Connes amenable.*

(ii) *$\mathcal{A} \times_\Theta \mathcal{B}$ is $(0, \theta)$-module Connes amenable if and only if $\mathcal{B}$ is θ-module Connes amenable.*

Proof. (i) Let $\mathcal{A}$ be ψ-module Connes amenable. Suppose that E is a symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module and $D_\mathcal{U} : \mathcal{A} \to E$ is a module ψ-derivation. For extend $D_\mathcal{U}$ we define $\tilde{D_\mathcal{U}} : \mathcal{A} \times_\Theta \mathcal{B} \to E$, $\tilde{D_\mathcal{U}}(a \otimes b) = D_\mathcal{U}(a).b + a.D_\mathcal{U}(b)$. It is clear that E is a symmetric normal Banach $\mathcal{A} \times_\Theta \mathcal{B}$-$\mathcal{U}$-module. We show that $\tilde{D_\mathcal{U}}$ is inner.

(ii) We prove this part by define the concept of invariant mean.

□

2.2. module Connes amenability of Banach algebras of module extension. In this subsection we study module Connes amenability and character module Connes amenability of dual Banach algebras of module extension. Suppose that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{X}$ be a normal Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$. The Banach algebra $\mathcal{A} \oplus \mathcal{X}$, the l^1-direct sum of a Banach algebra $\mathcal{A}$ and a nonzero Banach $\mathcal{A}$-bimodule $\mathcal{X}$ with the structure $\mathcal{A} \oplus \mathcal{X} = \{(a, x) : a \in \mathcal{A}, x \in \mathcal{X}\}$ and algebraic product and norm defined as

$$(a, x) \cdot (a', x') = (aa', ax' + xa'), \quad \|(a, x)\| = \|a\|_\mathcal{A} + \|x\|_\mathcal{X} \quad (a, a' \in \mathcal{A}, x, x' \in \mathcal{X})$$

is called a module extension Banach algebra. It is known that $\mathcal{A} \oplus \mathcal{X}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus_\infty \mathcal{X}_*$, where $\oplus_\infty$ denotes l_∞-direct sum of Banach $\mathcal{A}$-modules. For dual and the second dual of $\mathcal{A} \oplus \mathcal{X}$ we have $(\mathcal{A} \oplus \mathcal{X})^* = \mathcal{A}^* \oplus_\infty \mathcal{X}^*$ and $(\mathcal{A} \oplus \mathcal{X})^{**} = \mathcal{A}^{**} \oplus \mathcal{X}^{**}$, respectively. According to [2, pp. 27 and 28], $\mathcal{X}^{**}$ is Banach $\mathcal{A}^{**}$-bimodule, where $\mathcal{A}^{**}$ is equipped with the first Arens product.

We write $B(\mathcal{A}, \mathcal{A}^*)$ for Banach space of bounded linear maps from $\mathcal{A}$ to $\mathcal{A}^*$. Also, it is known that $B(\mathcal{A}, \mathcal{A}^*)$ is a Banach $\mathcal{A}$-bimodule with module multiplications via

$$\langle a.T, b\rangle = \langle T, ba\rangle, \qquad \langle T.a, b\rangle = \langle T, ab\rangle \qquad (a, b \in \mathcal{A}, T \in B(\mathcal{A}, \mathcal{A}^*)).$$

It is standard that $B(\mathcal{A}, \mathcal{A}^*) = (\mathcal{A}\hat{\otimes}\mathcal{A})^*$ where $\langle T, x \otimes y\rangle = \langle T(y), x\rangle$ for all $x, y \in \mathcal{A}$. This is equivalent to the following short exact sequence splits:

$$0 \longrightarrow \mathcal{A}^* \overset{\longrightarrow}{\underset{\leftarrow--}{}} B(\mathcal{A}, \mathcal{A}^*) \longrightarrow B(\mathcal{A}, \mathcal{A}^*)/\Delta^*_\mathcal{A}(\mathcal{A}^*) \longrightarrow 0$$

In other words, there exists a bounded bimodule homomorphism P from $B(\mathcal{A}, \mathcal{A}^*)$ to $\mathcal{A}_*$ such that $P \circ \Delta^*_\mathcal{A} = I_{\mathcal{A}^*}$ where $\Delta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is multiplication operator. In [3], it is proved that an unital dual Banach algebra $\mathcal{A}$ with predual $\mathcal{A}_*$ is Connes amenable if and only if the following short exact sequence splits:

$$0 \longrightarrow \mathcal{A}_* \overset{\longrightarrow}{\underset{\leftarrow--}{}} \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*) \longrightarrow \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)/\Delta^*_\mathcal{A}(\mathcal{A}_*) \longrightarrow 0$$

Theorem 2.5. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and let $\mathcal{X}$ be a normal Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$ that is pseudo-unital. Then $\mathcal{A} \oplus \mathcal{X}$ is module Connes amenable if and only if $\mathcal{A}$ is module Connes amenable.*

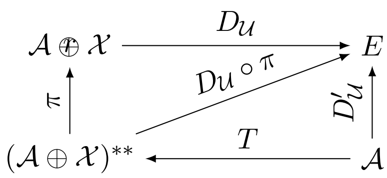
We know that $\Delta(\mathcal{A} \oplus \mathcal{X}) = \Delta(\mathcal{A}) \times \{0\}$. It is shown that if $\mathcal{A} \oplus \mathcal{X}$ is $(\psi, 0)$-amenable then $\mathcal{A}$ is ψ-amenable and the converse also holds in the case where $\mathcal{X}A = 0$. It would be interesting to know whether the result extends to $(\psi, 0)$-module Connes amenability. The following theorem is an analog of [5, Theorem 1.1].

Theorem 2.6. *Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and $\psi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. Let $\mathcal{X}$ be a normal Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$. Then the following holds:*

(i) *If $\mathcal{A} \oplus \mathcal{X}$ is $(\psi, 0)$-module Connes amenable, then $\mathcal{A}$ is ψ-module Connes amenable.*
(ii) *If $\mathcal{X}\mathcal{A} = 0$ and $\mathcal{A}$ is ψ-module Connes amenable, then $\mathcal{A} \oplus \mathcal{X}$ is $(\psi, 0)$-module Connes amenable.*

Proof. (i) Let E be a symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module. Therefore, E is $\mathcal{A}\oplus\mathcal{X}$-$\mathcal{U}$-module. Let $D_{\mathcal{U}} : \mathcal{A} \to E$ be a bounded $weak^*$-continuous ψ-derivation. Let $P : \mathcal{A} \oplus \mathcal{X} \to \mathcal{A}$ be the projection map. It is known that $P^*(a.f) = a.P^*(f)$ for all $f \in \mathcal{A}_*, a \in \mathcal{A}$. We put $\tilde{D_{\mathcal{U}}} = D_{\mathcal{U}} \circ P : \mathcal{A} \oplus \mathcal{X} \to E$. It is clear that $\tilde{D_{\mathcal{U}}}$ is module $(\psi, 0)$-derivation. We show that $D_{\mathcal{U}}$ is inner.

(ii) Let E be a symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module. Since $\mathcal{A}$ is a normal Banach $\mathcal{A}$-module then E is a symmetric Banach $\mathcal{A} \oplus \mathcal{X}$-$\mathcal{U}$-module. Let $D'_{\mathcal{U}} : \mathcal{A} \to E$ be a bounded $weak^*$-continuous module ψ-derivation. Suppose that $D_{\mathcal{U}} : \mathcal{A} \oplus \mathcal{X} \to E$ is a bounded $weak^*$-continuous module $(\psi, 0)$-derivation. Let $D^*_{\mathcal{U}} : E^* \to (\mathcal{A} \oplus \mathcal{X})^*$ denotes the adjoint of $D_{\mathcal{U}}$. Then $D^*_{\mathcal{U}}$ maps the predual E_* of E^* into $\mathcal{A}_* \oplus \mathcal{X}_*$. Let $\pi : (\mathcal{A} \oplus \mathcal{X})^{**} \to (\mathcal{A} \oplus \mathcal{X})$ be the Dixmier projection. Now consider the following commutative diagram,



By hypothesis, $D'_{\mathcal{U}} = D_{\mathcal{U}} \circ \pi \circ T : \mathcal{A} \to E$ is inner. Now we must show that $D_{\mathcal{U}}$ is inner. □

3. Conclusion

By using module Connes amenability of primary Banach algebras, we characterize the module Connes amenability of Banach algebras of module extension and Θ-Lau product dual Banach algebras.

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