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φ-CONNES MODULE AMENABILITY OF DUAL BANACH ALGEBRAS

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Abstract. In this paper, we define φ-Connes module amenability of a dual Banach algebra $\mathcal{A}$, where φ is a bounded module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ that is w_{k^*}-continuous. We are mainly concerned with the study of φ-module normal, virtual diagonals. We show that if S is a weakly cancellative and S is an inverse semigroup with subsemigroup E of idempotents, χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is w_{k^*}-continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ-Connes module amenable, then it has a χ-module normal, virtual diagonal. In the case $\chi = id$, the converse also holds.

1. Introduction

Connes amenable dual Banach algebras were introduced by Runde in [19]. In [20], Runde showed that if a Banach algebra is Connes amenable, it has a normal, virtual diagonal. The interest in normal, virtual diagonals arises from the fact that for a von Neumann algebra $\mathcal{A}$, Connes amenability of $\mathcal{A}$ is completely determined by the existence of a normal, virtual diagonal. As noticed by Runde, the notion of a normal, virtual diagonal adapts naturally to the context of general dual Banach algebras. In [21], it is shown that $M(G)$, the measure algebra of a locally compact group G, is Connes amenable if and only if it has a normal, virtual diagonal.

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In [1], Amini introduced the concept of module amenability for Banach algebras, and proved that when S is an inverse semigroup with subsemigroup E of idempotents, then $l^1(S)$ as a Banach module over $\mathcal{U} = l^1(E)$ is module amenable if and only if S is amenable. Also, in [2], it is shown that $l^1(S)^{**}$ is $l^1(E)$-module amenable if and only if an appropriate group homomorphic image of S is finite. We may refer the reader e.g. to [1, 2, 3, 4, 5, 16], for extensive treatments of various notions of module amenability.
All of these concepts generalized the earlier concept of amenability for Banach algebras introduced by Johnson [12]. Recently, the authors have introduced the ϕ-version of Connes amenability of dual Banach algebra $\mathcal{A}$ that ϕ is a homomorphism from $\mathcal{A}$ onto $\mathbb{C}$ that lies in $\mathcal{A}_*$[11]. Let $\mathcal{A}$ be a dual Banach algebra with a compatible action of a Banach algebra $\mathcal{U}$ and φ be a bounded module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ that is w_{k^*}-continuous. In this paper, we introduce the concept of φ-Connes module amenability for $\mathcal{A}$ and give a characterization of φ-Connes module amenability in terms of φ-modul normal virtual diagonals. In particular, we show that if χ is a bounded module homomorphism from $l^1(S)$ to $l^1(S)$ that is w_{k^*}-continuous and $l^1(S)$ as a Banach module over $l^1(E)$ is χ-Connes module amenable, then it has a χ-module normal virtual diagonal. In the case $\chi = id$, the converse also holds, restoring [21, Theorem 1] for the case of measure algebra of a discrete group.

2. Main results

Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule with compatible actions, that is

$$\alpha.(ab) = (\alpha.a).b,\ (\alpha\beta).a = \alpha.(\beta.a)\ (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let E be a dual Banach $\mathcal{A}$-bimodule. E is called *normal* if for each $x \in E$, the maps

$$\mathcal{A} \to E,\ a \to \begin{cases} a.x \\ x.a \end{cases}$$

are w_{k^*}- continuous. If moreover E is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}$, $\alpha \in \mathcal{U}$ and $x \in E$

$$\alpha.(a.x) = (\alpha.a).x,\ (a.\alpha).x = a.(\alpha.x),\ (\alpha.x).a = \alpha.(x.a),$$

then E is called a *normal Banach left $\mathcal{A}-\mathcal{U}$-module*. Similarly for the right and two sided actions. Also, E is called *commutative*, if

$$\alpha.x = x.\alpha\ (\alpha \in \mathcal{U}, x \in E).$$

A module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is a map $\varphi : \mathcal{A} \to \mathcal{A}$ with

$$\varphi(a+b) = \varphi(a) + \varphi(b),\ \varphi(ab) = \varphi(a)\varphi(b)$$
$$\varphi(\alpha.a) = \alpha.\varphi(a),\ \varphi(a.\alpha) = \varphi(a).\alpha\ (a, b \in \mathcal{A}, \alpha \in \mathcal{U}).$$

Since $\mathcal{A}$ is a dual Banach algebra, then multiplication in $\mathcal{A}$ is w_{k^*}-continuous. Consider $\mathcal{A}$ as dual $\mathcal{A}$-module with predual $\mathcal{A}_*$. So we shall suppose that $\mathcal{A}$ takes w_{k^*}-topology. $\mathcal{HOM}_{w_{k^*}}(\mathcal{A})$ will denote the space of all bounded module homomorphism that is w_{k^*}-continuous.

A bounded map $D : \mathcal{A} \to E$ is called a module φ-derivation if

$$D(a \pm b) = D(a) \pm D(b), D(ab) = D(a).\varphi(b) + \varphi(a).D(b)$$
$$D(\alpha.a) = \alpha.D(a), D(a.\alpha) = D(a).\alpha\ (a, b \in \mathcal{A}, \alpha \in \mathcal{U}).$$

When E is commutative, each $x \in E$ defines a module φ-derivation

$$D_x(a) = \varphi(a).\ x - x.\ \varphi(a)\ (a \in \mathcal{A}).$$

Derivations of this form are called *inner module φ-derivation.*

Definition 2.1. Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. $\mathcal{A}$ is called φ-Connes module amenable if for any commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-module E, each w_{k^*}- continuous module φ-derivation $D : \mathcal{A} \to E$ is inner.

Recall that if φ is identity map on $\mathcal{A}$, then id-Connes module amenability is called Connes module amenability. Also, by the proof of [1, Proposition 2.1], Connes amenability of $\mathcal{A}$ implies its Connes module amenability in the case where $\mathcal{U}$ has a bounded approximate identity for $\mathcal{A}$. Example 9 shows that the converse is false. Hence Connes module amenability is weaker than Connes amenability. Throughout this paper, $\mathcal{A}$ is a Banach algebra that is a Banach $\mathcal{U}$-module.

Theorem 2.2. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. If φ is an epimorphism and $\mathcal{A}$ is φ-Connes module amenable, then $\mathcal{A}$ is Connes-module amenable.*

Proof. Let E be a commutative normal Banach $\mathcal{A} - \mathcal{U}$-module and $D : \mathcal{A} \to E$ be a w_{k^*}-continuous module derivation. Set $d = D \circ \varphi$. The mapping $d : \mathcal{A} \to E$ is a module φ-derivation. Since $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$, then d is w_{k^*}-continuous. Thus there exists $f \in E$ such that $d(a) = f.\ \varphi(a) - \varphi(a).\ f$ for all $a \in \mathcal{A}$. Let $b \in \mathcal{A}$, there exists $a \in \mathcal{A}$ such that $\varphi(a) = b$. Hence

$$D(b) = D(\varphi(a)) = d(a) = f.\ \varphi(a) - \varphi(a).\ f = f.\ b - b.\ f.$$

This shows that $\mathcal{A}$ is Connes-module amenable. ∎

Theorem 2.3. *Let $\mathcal{A}$ be a dual Arens regular Banach algebra and $\varphi \in \mathcal{H}OM_{w_{k^*}}(\mathcal{A})$. Then the following are equivalent:*
(i) *$\mathcal{A}$ is φ- Connes module amenable.*
(ii) *$\mathcal{A}^{**}$ is φ^{**}-Connes module amenable.*

Proof. (i)⇒(ii) Let E be a commutative normal Banach $\mathcal{A}^{**}$–$\mathcal{U}$-module and $D : \mathcal{A}^{**} \to E$ be a w_{k^*}-continuous module φ^{**}-derivation. Let $\theta : \mathcal{A} \to \mathcal{A}^{**}$ be the canonical map. It is known that θ is w_{k^*}-continuous. Define a module action of $\mathcal{A}$ on E by letting $x \bullet a = x.\ \theta(a),\ a \bullet x = \theta(a).\ x$ $(a \in \mathcal{A}, x \in E)$. It can be shown that this module action is well defined and turns E into a normal Banach $\mathcal{A} - \mathcal{U}$-module. We define a derivation $\tilde{D} : \mathcal{A} \to E$ by letting $\tilde{D} = D \circ \theta$. Then we have

$$\begin{aligned}\tilde{D}(ab) = D \circ \theta(ab) &= D \circ \theta(a).\ \varphi^{**}(\theta(b)) + \varphi^{**}(\theta(a)).\ D \circ \theta(b)\\ &= D \circ \theta(a).\ \theta(\varphi(b)) + \theta(\varphi(a)).\ D \circ \theta(b)\\ &= D \circ \theta(a) \bullet\ \varphi(b) + \varphi(a) \bullet\ D \circ \theta(b)\\ &= \tilde{D}(a) \bullet\ \varphi(b) + \varphi(a) \bullet\ \tilde{D}(b).\end{aligned}$$

Thus $\tilde{D}$ is a module φ-derivation that is w_{k^*}-continuous. Since $\mathcal{A}$ is φ-Connes module amenable, then there exists $x \in E$ such that

$$\begin{aligned}\tilde{D}(a) &= D \circ \theta(a) = x \bullet\ \varphi(a) - \varphi(a) \bullet\ x\\ &= x.\ \theta(\varphi(a)) - \theta(\varphi(a)).\ x.\end{aligned}$$

Let $G \in \mathcal{A}^{**}$. As $\theta(\mathcal{A})$ is w_{k^*}-dense in $\mathcal{A}^{**}$, there exists a net $\{g_\alpha\}$ in $\mathcal{A}$ such that $\theta(g_\alpha) \to G$ in the w_{k^*}-topology. Also it is known that φ^{**} is w_{k^*}-continuous, then $\varphi^{**}(\theta(g_\alpha)) \to \varphi^{**}(G)$. Hence

$$\begin{aligned}D(G) = \lim_\alpha D \circ \theta(g_\alpha) &= \lim_\alpha x.\ \theta \circ \varphi(g_\alpha) - \theta \circ \varphi(g_\alpha).\ x\\ &= \lim_\alpha x.\ \varphi \circ \theta(g_\alpha) - \varphi \circ \theta(g_\alpha).\ x\\ &= x.\ \varphi^{**}(G) - \varphi^{**}(G).\ x\end{aligned}$$

(ii)⇒(i) Let E be a commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-module and $D : \mathcal{A} \to E$ be a w_{k^*}-continuous module φ-derivation. Let $\pi : (\mathcal{A}_*)^{***} \to (\mathcal{A}_*)^*$ by $\pi(F) = F\ |_{\theta(\mathcal{A}_*)}$ be the Dixmier projection. It is well known that the Dixmier projection from $\mathcal{A}^{**}$ onto $\mathcal{A}$ is a module homomorphism [14]. Then E is a Banach $\mathcal{A}^{**}$-$\mathcal{U}$-module with the bimodule multiplications

$$F \bullet x = \pi(F).\ x,\ \ x \bullet F = x.\ \pi(F)\ \ (x \in E, F \in \mathcal{A}^{**}).$$

It is routinely checked that E is a commutative normal Banach $\mathcal{A}^{**}$-$\mathcal{U}$-module. Now set $D\circ\pi : \mathcal{A}^{**} \to E$. We have

$$\begin{aligned} D\circ\pi(FG) &= D(\pi(F)\pi(G)) = Do\pi(F).\ \varphi\circ\pi(G) + \varphi\circ\pi(F).\ Do\pi(G) \\ &= D\circ\pi(F).\ \varphi^{**}\circ\pi(G) + \varphi^{**}\circ\pi(F).\ D\circ\pi(G) \\ &= D\circ\pi(F).\ \pi(\varphi^{**}(G)) + \pi(\varphi^{**}(F)).\ D\circ\pi(G) \\ &= D\circ\pi(F)\bullet\ \varphi^{**}(G) + \varphi^{**}(F)\bullet\ D\circ\pi(G). \end{aligned}$$

Since $\mathcal{A}^{**}$ is φ^{**}-Connes module amenable, then there exists $x \in E$ such that

$$\begin{aligned} Do\pi(F) = \varphi^{**}(F)\bullet\ x - x\bullet\ \varphi^{**}(F) &= \pi(\varphi^{**}(F)).\ x - x.\ \pi(\varphi^{**}(F)) \\ &= \varphi^{**}(\pi(F)).\ x - x.\ \varphi^{**}(\pi(F)). \end{aligned}$$

Therefore $D(a) = \varphi(a).\ x - x.\ \varphi(a)$ for all $a \in \mathcal{A}$, and hence D is inner. ☐

Theorem 2.4. *Let $\mathcal{A}$ be a commutative dual Banach algebra and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. If $\mathcal{A}$ is φ-Connes module amenable, then $\mathcal{A}$ has a bounded approximate identity for $\varphi(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ be a commutative Banach $\mathcal{A}$-$\mathcal{U}$-module whose underlying space is $\mathcal{A}$, but on which $\mathcal{A}$ acts via

$$a.\ x := ax,\ \ x.\ a := 0\ (a \in \mathcal{A}, x \in \mathcal{A}).$$

Let $I : \mathcal{A} \to \mathcal{A}$ be the identity map. It is easy to see that $I\circ\varphi$ is a module φ-derivation. Since $\mathcal{A}$ is φ-Connes module amenable, there exists $e \in \mathcal{A}$ such that

$$\begin{aligned} I\circ\varphi(a) &= \varphi(a).\ e - e.\ \varphi(a) \\ \varphi(a) &= \varphi(a).\ e. \end{aligned}$$

The element e has the desired properties. ☐

Theorem 2.5. *Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. If $\mathcal{A}$ is φ-Connes module amenable, then $\mathcal{A}$ is $\lambda\circ\varphi$-Connes module amenable for any $\lambda \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$.*

Proof. Let E be a commutative normal Banach $\mathcal{A}-\mathcal{U}$-module and $D : \mathcal{A} \to E$ be a module $\lambda\circ\varphi$-derivation that is w_{k^*}- continuous. If E is equipped with the module operation by

$$a\bullet x = \lambda(a).\ x,\ \ x\bullet a = x.\ \lambda(a),\ \ (a \in \mathcal{A}, x \in E)$$

then E becomes a commutative normal Banach $\mathcal{A}-\mathcal{U}$-module. We have

$$\begin{aligned} D(ab) &= D(a).\ \lambda\circ\varphi(b) + \lambda\circ\varphi(a).\ D(b) \\ &= D(a)\bullet\varphi(b) + \varphi(a)\bullet D(b). \end{aligned}$$

Thus, there exists $f \in E$ such that

$$D(a) = f \bullet \varphi(a) - \varphi(a) \bullet f = f.\ \lambda \circ \varphi(a) - \lambda \circ \varphi(a).\ f\ (a \in \mathcal{A}).$$

This shows that D is inner. □

Theorem 2.6. *Let $\mathcal{A}$ be a unital dual Banach algebra and also $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. Then $\mathcal{A}$ is φ-Connes module amenable if and only if for any unital commutative Banach $\mathcal{A}$-$\mathcal{U}$-module E, each module φ-derivation $D : \mathcal{A} \longrightarrow E$ is inner.*

Proof. Let E be a commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-bimodule with predual E_*, and consider $l : E \longrightarrow E$ and $r : E \longrightarrow E$ by $l(x) = e_{\mathcal{A}}x$ and $r(x) = xe_{\mathcal{A}}$. put $E_1 = (id - l) \circ r(E)$, $E_2 = (id - r) \circ l(E)$, $E_3 = (id - l) \circ (id - r)(E)$ and $E_4 = l \circ r(E)$. The verification that $E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ is routine. It is the diect sum of E_i for $i = 1, 2, 3, 4$. Then E_1 is equipped with the module operation by

$$(x - e_{\mathcal{A}}x).\ a = x.\ a - e_{\mathcal{A}}x.\ a,\ \ a(x - e_{\mathcal{A}}x) = a.\ x - a.\ e_{\mathcal{A}}.x = 0$$

It is easy to see that E_1 is a commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-bimodule by predual $(1 - e_{\mathcal{A}}).E_*.e_{\mathcal{A}}$. Let $\pi_1 : E \longrightarrow E_1$ be the projection map. Then $\pi_1 \circ D$ is a module φ-derivation from $\mathcal{A}$ to E_1 that is w_{k^*}- continuous. Since $\mathcal{A}$ has a left zero action on E_1, then we have

$$\begin{aligned} \pi_1 \circ D(a) &= \pi_1 \circ D(e_{\mathcal{A}}.a) = \pi_1 \circ D(e_{\mathcal{A}}).\varphi(a) + \varphi(e_{\mathcal{A}}).\pi_1 \circ D(a) \\ &= \pi_1 \circ D(e_{\mathcal{A}}).\varphi(a) = \pi_1 \circ D(e_{\mathcal{A}}).\varphi(a) - \varphi(a).\pi_1 \circ D(e_{\mathcal{A}}) \end{aligned}$$

Also, a routine verification shows that $\pi_2 o D = ad_{\pi_2 \circ D(e_{\mathcal{A}})}$ and $\pi_3 \circ D = 0$.

Now, let $\pi_4 \circ D : \mathcal{A} \longrightarrow E_4$. It is obvious that $\pi_4 \circ D$ is a module φ-derivation. We can show that E_4 is a commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-bimodule with predual $e_{\mathcal{A}}.E_*.e_{\mathcal{A}}$. By our assumption, $\pi_4 \circ D$ is inner. □

Let $\mathcal{A}$ and $\mathcal{U}$ are dual Banach algebras. Let $\mathcal{A}$ be a dual Banach $\mathcal{U}$-module and $\mathcal{A}\hat{\otimes}\mathcal{A}$ denote the projective tensor product of $\mathcal{A}$ and $\mathcal{A}$. Let $\mathcal{A}_* \otimes_w \mathcal{A}_*$ be the injective tensor product of $\mathcal{A}_*$ with itself. Then we have a canonical map from $\mathcal{A}\hat{\otimes}\mathcal{A}$ into $(\mathcal{A}_* \otimes_w \mathcal{A}_*)^*$ which has closed range if $\mathcal{A}$ has the bounded approximation property. For more details, see [18]. Let I be the closed ideal of $\mathcal{A}\hat{\otimes}\mathcal{A}$ generated by elements of the form $\alpha.(a \otimes b) - (a \otimes b).\alpha$, for $a, b \in \mathcal{A}$ and $\alpha \in \mathcal{U}$. $\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A}$ is defined to be the quitiont Banach space $\dfrac{\mathcal{A}\hat{\otimes}\mathcal{A}}{I}$[15]. Let J be the closed ideal of $\mathcal{A}$ generated by elements of the form $(\alpha.a).b - a.(b.\alpha)$. Since J is weak*-closed, then the quotient algebra $\dfrac{\mathcal{A}}{J}$ is again dual with predual

${}^{\perp}J = \{\phi \in \mathcal{A}_* : \langle \phi, a\rangle = 0 \text{ for all } a \in J\}$. Moreover, $\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A} \cong \dfrac{\mathcal{A}\hat{\otimes}\mathcal{A}}{I}$ and $\dfrac{\mathcal{A}}{J}$ could be regarded as a Banach $\mathcal{A}$-$\mathcal{U}$-module. Let $\mathcal{L}^2_{w^*}(\dfrac{\mathcal{A}}{J}, \mathbb{C})$ denote the separately w_{k^*}-continuous 2-linear maps from $\dfrac{\mathcal{A}}{J} \times \dfrac{\mathcal{A}}{J}$ to $\mathbb{C}$. Note that the dual Banach $\mathcal{A}$-$\mathcal{U}$-module $\mathcal{L}^2_{w^*}(\dfrac{\mathcal{A}}{J}, \mathbb{C})$ need not be normal. Let $\tilde{w} : \mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A} \to \dfrac{\mathcal{A}}{J}$ be the multiplication operator, $\tilde{w}(a \otimes b + I) = ab + J$. Since the quotient map is continuous and open, then it is immediate that $\tilde{w}^*$ maps ${}^{\perp}J$ into $\mathcal{L}^2_{w^*}(\dfrac{\mathcal{A}}{J}, \mathbb{C})$. It follows that $\tilde{w}^{**}$ drops to an $\mathcal{A}$-$\mathcal{U}$-module homomorphism $\tilde{w}^{**} : \mathcal{L}^2_{w^*}(\dfrac{\mathcal{A}}{J}, \mathbb{C})^* \to \dfrac{\mathcal{A}}{J}$. Recall a few definitions from [10](with a different notation, however). Given $F \in \mathcal{L}^2_{w^*}(\dfrac{\mathcal{A}}{J}, \mathbb{C})$ and $M \in \mathcal{L}^2_{w^*}(\dfrac{\mathcal{A}}{J}, \mathbb{C})^*$, we put

$$\langle M,\ F\rangle = \int F dM =: \int_{\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A}} F(a+J, b+J) dM(a+J, b+J).$$

More generally, let E be a dual Banach space and let $F : \dfrac{\mathcal{A}}{J} \times \dfrac{\mathcal{A}}{J} \to E$ be a bilinear map such that $a + J \to F(a + J, b + J)$ and $b + J \to F(a + J, b + J)$ are w_{k^*}- continuous. We define $\int F dM \in E$ by

$$\langle \int F dM,\ x\rangle = \int \langle F(a+J, b+J), x\rangle dM(a+J, b+J),$$

where $a, b \in \mathcal{A}, x \in E_*$. Let $\varphi \in \mathcal{HOM}_{w_{k^*}}(\mathcal{A})$ such that $\varphi(J) \subseteq J$. Then the map $\tilde{\varphi} : \dfrac{\mathcal{A}}{J} \to \dfrac{\mathcal{A}}{J}$ by $\tilde{\varphi}(a+J) = \varphi(a)+J$ could be considered as an element of $\mathcal{HOM}_{w_{k^*}}(\dfrac{\mathcal{A}}{J})$.

Definition 2.7. Let $\mathcal{A}$ be a dual Banach algebra. An element $M \in \mathcal{L}^2_{w^*}(\dfrac{\mathcal{A}}{J}, \mathbb{C})^*$ is called a φ-module normal virtual diagonal for $\mathcal{A}$ if $\tilde{w}^{**}(M)$ is an identity for $\dfrac{\varphi(\mathcal{A})}{J}$ and

$$M.\ \tilde{\varphi}(c+J) = \tilde{\varphi}(c+J).\ M\ (c \in \mathcal{A}).$$

Note that with the above notation $M.\ (c + J) = (c + J).\ M$ is equivalent to

$$\int F(ca+J, b+J) dM(a+J, b+J) = \int F(a+J, bc+J) dM(a+J, b+J).$$

Theorem 2.8. *Let $\mathcal{A}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}$ be a unital dual Banach $\mathcal{U}$- module and let $\mathcal{A}$ has an id-module normal virtual diagonal. Then $\mathcal{A}$ is id-Connes module amenable.*

Proof. Let E be a commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-module. We first note that $\mathcal{A}$ has an identity. From Theorem 5, it is therefore sufficient for $\mathcal{A}$ to be *id*-Connes module amenable that we suppose that E is unital. Let $D : \mathcal{A} \to E$ be a module derivation that is w_{k^*}- continuous. It is straightforward to see that E is a normal Banach $\frac{\mathcal{A}}{J}$-$\mathcal{U}$-module. Let $E = (E_*)^*$. Since E is commutative, then $D = 0$ on J. Thus we have $\tilde{D} : \frac{\mathcal{A}}{J} \to E,\ \tilde{D}(a + J) := D(a)\ (a \in \mathcal{A})$. To each $x \in E_*$, there corresponds $V_x : \frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J} \to \mathbb{C}$ via $V_x(a + J, b + J) = \langle x, (a + J)\tilde{D}(b + J)\rangle (a, b \in \mathcal{A})$. It is routinely checked that $V_x \in \mathcal{L}^2_{w^*}(\frac{\mathcal{A}}{J}, \mathbb{C})$. For each $a, b \in \mathcal{A}$ and $a_* \in \mathcal{A}_*$ we have

$$
\begin{aligned}
\langle \int ab + JdM, a_* + J^\perp\rangle &= \langle \int \tilde{w}(a \otimes b + I)dM, a_* + J^\perp\rangle \\
&= \int \langle \tilde{w}(a \otimes b + I), a_* + J^\perp\rangle dM \\
&= \int \langle a \otimes b + I, \tilde{w}^*(a_* + J^\perp)\rangle dM \\
&= \langle \int a \otimes b + IdM, \tilde{w}^*(a_* + J^\perp)\rangle \\
&= \langle M, \tilde{w}^*(a_* + J^\perp)\rangle = \langle \tilde{w}^{**}(M), a_* + J^\perp\rangle,
\end{aligned}
$$

Now, put $f(x) = \langle M,\ v_x\rangle (x \in E_*)$. Let $c \in \mathcal{A}$. We have

$$
\begin{aligned}
&\langle \quad (c + J).\ f - f.\ (c + J),\ x\rangle \\
&= \langle f, x.\ (c + J) - (c + J).\ x\rangle \\
&= \langle M,\ V_{x.\ (c+J)-(c+J).\ x}\rangle \\
&= \int V_{x.\ (c+J)-(c+J).\ x}(a + J, b + J)dM \\
&= \int \langle x.\ (c + J) - (c + J).\ x,\ (a + J)\tilde{D}(b + J)\rangle dM \\
&= \int \langle x,\ (c + J)(a + J)\tilde{D}(b + J) - (a + J)\tilde{D}(b + J)(c + J)\rangle dM \\
&= \int \langle x,\ (ca + J)\tilde{D}(b + J) - (a + J)\tilde{D}(b + J)(c + J)\rangle dM,
\end{aligned}
$$

and so

$$\begin{aligned}
\langle &\quad (c+J).\ f - f.\ (c+J),\ x\rangle \\
= &\int \langle x,\ (a+J)\tilde{D}(bc+J) - (a+J)\tilde{D}(b+J)(c+J)\rangle dM \\
= &\int \langle x,\ (a+J)\tilde{D}(b+J)(c+J) + (a+J)(b+J)\tilde{D}(c+J) \\
- &\quad (a+J)\tilde{D}(b+J)(c+J)\rangle dM \\
= &\int \langle (a+J)(b+J)\tilde{D}(c+J),\ x\rangle dM \\
= &\int \langle (ab+J)\tilde{D}(c+J),\ x\rangle dM \\
= &\int \langle (ab+J),\ x\rangle dM.\tilde{D}(c+J) \\
= &\quad \langle \tilde{w}^{**}(M).\ \tilde{D}(c+J),\ x\rangle.
\end{aligned}$$

All in all, $D(c) = c.\ f - f.\ c$ holds. □

Let $\mathcal{A}$ be a commutative Banach $\mathcal{U}$-bimodule. Consider $\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A}$ with the product specified by $(a\otimes b)(c\otimes d) = ac\otimes bd$. Let $\varphi\otimes\varphi$ denote the element of $\mathcal{HOM}_{w_{k^*}}(\mathcal{A}\hat{\otimes}\mathcal{A})$ satisfying $\varphi\otimes\varphi(a\otimes b) = \varphi(a)\otimes\varphi(b)$ for all $a,\ b\in\mathcal{A}$. $\varphi\otimes\varphi$ induces a map $\varphi\otimes_{\mathcal{U}}\varphi\in\mathcal{HOM}_{w_{k^*}}(\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A})$ with $\varphi\otimes_{\mathcal{U}}\varphi(a\otimes b) = \varphi(a)\otimes\varphi(b) + I$ [7].

Theorem 2.9. *Let $\mathcal{A}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}$ be a unital dual Banach $\mathcal{U}$- module and let $\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A}$ be a dual Banach algebra and $\varphi\in\mathcal{HOM}_{w_{k^*}}(\mathcal{A})$. If $\mathcal{A}$ is φ-Connes module amenable, then $\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A}$ is $\varphi\otimes_{\mathcal{U}}\varphi$-Connes module amenable.*

Proof. Let E be a commutative normal Banach $\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A}$-$\mathcal{U}$-module and $\hat{D}:\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A}\to E$ be a module $\varphi\otimes_{\mathcal{U}}\varphi$-derivation that is w_{k^*}- continuous. Consider the quotient map $\pi:\mathcal{A}\hat{\otimes}\mathcal{A}\to\mathcal{A}\hat{\otimes}_{\mathcal{U}}\mathcal{A}$. Define

$$(a\otimes b).\ x = \pi(a\otimes b)\ominus\ x,\ x.\ (a\otimes b) = x\ominus\ \pi(a\otimes b)\ (a,b\in\mathcal{A},x\in E)$$

Since π is w_{k^*}- continuous, then E is a normal Banach $\mathcal{A}\hat{\otimes}\mathcal{A}$-$\mathcal{U}$-module. Put $\hat{D}\circ\pi:\mathcal{A}\hat{\otimes}\mathcal{A}\to E$. It is easy to see that $\hat{D}\circ\pi$ is a module $\varphi\otimes\varphi$-derivation that is w_{k^*}- continuous. If $\hat{D}\circ\pi$ is inner, then $\hat{D}$ is inner. Therefore in the following we prove that $D=\hat{D}\circ\pi$ is inner. For with $e_{\mathcal{A}}$ an identity for $\mathcal{A}$ we define

$$a\ \triangle\ \ x = (a\otimes e_{\mathcal{A}}).\ x,\ x\ \triangle\ \ a = x.\ (a\otimes e_{\mathcal{A}})\ (a\in\mathcal{A},\ x\in E).$$

For $a \in \mathcal{A},\ x \in E$ and $\alpha \in \mathcal{U}$, we get

$$\begin{aligned} a \vartriangle \ (\alpha.\ x) - (a.\ \alpha) \vartriangle \ x &= (a \otimes e_{\mathcal{A}}).\ (\alpha.\ x) - (a.\ \alpha \otimes e_{\mathcal{A}}).\ x \\ &= (a \otimes e_{\mathcal{A}}).\ (\alpha.\ x) - (\alpha.\ a \otimes e_{\mathcal{A}}).\ x \\ &= (a \otimes e_{\mathcal{A}}).\ (\alpha.\ x) - (\alpha.\ (a \otimes e_{\mathcal{A}})).\ x \\ &= (a \otimes e_{\mathcal{A}}).\ (\alpha.\ x) - ((a \otimes e_{\mathcal{A}}).\ \alpha).\ x \\ &= (a \otimes e_{\mathcal{A}}).\ (\alpha.\ x) - (a \otimes e_{\mathcal{A}}).(\alpha.\ x) = 0 \end{aligned}$$

and the same for the right or two-sided actions. So E is a commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-bimodule. Put $D_{\mathcal{A}} : \mathcal{A} \to E$, $D_{\mathcal{A}}(a) = D(a \otimes e_{\mathcal{A}})$, then

$$\begin{aligned} D_{\mathcal{A}}(ab) &= D(ab \otimes e_{\mathcal{A}}) \\ &= D(a \otimes e_{\mathcal{A}}).\ \varphi \otimes \varphi(b \otimes e_{\mathcal{A}}) + \varphi \otimes \varphi(a \otimes e_{\mathcal{A}}).\ D(b \otimes e_{\mathcal{A}}) \\ &= D_{\mathcal{A}}(a) \vartriangle \ \varphi(b) + \ \varphi(a) \vartriangle \ D_{\mathcal{A}}(b). \end{aligned}$$

Since $\mathcal{A}$ is φ-Connes module amenable, there is $u \in E$ such that $D_{\mathcal{A}} = ad_u$. Therefore, $\tilde{D} = D - ad_u$ vanishes on $\mathcal{A} \otimes e_{\mathcal{A}}$. Setting

$$a\triangledown\ x = (e_{\mathcal{A}} \otimes a).\ x,\ \ x\triangledown\ a = x.\ (e_{\mathcal{A}} \otimes a)\ (a \in \mathcal{A},\ x \in E)$$

makes E into an $\mathcal{A}$-$\mathcal{U}$-bimodule. Let us now, $D'_{\mathcal{A}}(a) = \tilde{D}(e_{\mathcal{A}} \otimes a)(a \in \mathcal{A})$. Set $K = \{e \in E_* : \langle \tilde{D}(e_{\mathcal{A}} \otimes a),\ e\rangle = 0\}$. Since $\tilde{D}$ is w_{k^*}-continuous, by a similar argument of [17, Theorem 4.9] we have $(\frac{E_*}{K})^* = \overline{\tilde{D}(e_{\mathcal{A}} \otimes a)}^{w_k^*}$. Further, $\overline{\tilde{D}(e_{\mathcal{A}} \otimes a)}^{w_k^*}$ is a w_{k^*}-closed submodule of E. All in all $\overline{\tilde{D}(e_{\mathcal{A}} \otimes a)}^{w_k^*}$ is a commutative normal Banach $\mathcal{A}$-$\mathcal{U}$-module. Then there is $v \in \overline{\tilde{D}(e_{\mathcal{A}} \otimes a)}^{w_k^*}$ such that

$$\tilde{D}(e_{\mathcal{A}}\otimes a) = D'_{\mathcal{A}}(a) = \varphi(a)\triangledown\nu - \nu\triangledown\varphi(a) = \varphi\otimes\varphi(e_{\mathcal{A}}\otimes a).\ v - v.\ \varphi\otimes\varphi(e_{\mathcal{A}}\otimes a)$$

and $\tilde{D} - ad_v|_{(e_{\mathcal{A}}\otimes\mathcal{A})} = \{0\}$. Consequently $\tilde{D} - ad_v = D - ad_u - ad_v$ vanishes on $\mathcal{A}\hat{\otimes}\mathcal{A}$. This complete the proof. □

3. χ-Connes Module amenability of semigroup algebras

A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotent elements of S is denoted by E. For $s \in S$, we define $L_s, R_s : S \to S$ by $L_s(t) = st, R_s(t) = ts,\ (t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative.

Before turning our result, we note that if S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$[8].

In Theorem 2.8 it is shown that if a unital Banach algebra $\mathcal{A}$ has an id-module normal virtual diagonal, then $\mathcal{A}$ is id-Connes module amenable. It would be interesting to know that the converse holds for inverse semigroup algebra $l^1(S)$.
For an inverse semigroup S, we consider an equivalence relation on S where $s \sim t$ if and only if there is $e \in E$ such that $se = te$. The quotient semigroup $S_G = \dfrac{S}{\sim}$ is a group [13]. It is easy to see that E is a commutative subsemigroup of S. Therefore, $l^1(S)$ is a Banach $l^1(E)$-module with compatible canonical actions. Let $l^1(E)$ acts on $l^1(S)$ by the multiplication from right and trivially from left, that is

$$\delta_e.\delta_s = \delta_s,\ \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e\ (s \in S, e \in E).$$

With above notation, $l^1(S_G)$ is a quotient of $l^1(S)$ and so the above action of $l^1(E)$ on $l^1(S)$ lifts to an action of $l^1(E)$ on $l^1(S_G)$, making it a Banach $l^1(E)$-module [1].

Theorem 3.1. *Let S be a weakly cancellative semigroup. Let S be an inverse semigroup with idempotents E, let $l^1(S)$ be a Banach $l^1(E)$-module and let $\chi \in \mathcal{HOM}_{w_{k^*}}(l^1(S))$. If $l^1(S)$ is χ-Connes module amenable, then $l^1(S)$ has a χ-module normal virtual diagonal.*

Proof. Let $\pi : S \to S_G$ be the quotient map. By [1, Lemma 3.2], we define a bimodule action of $l^1(S)$ on $l^\infty(S_G)$ by

$$\delta_s.\ x = \delta_{\pi(s)} * x,\ x.\ \delta_s = x * \delta_{\pi(s)}\ (s \in S,\ x \in l^\infty(S_G)).$$

Since $c_0(S_G)$ is an introverted subspace of $l^\infty(S_G)$ [9], then $l^\infty(S_G)^*$ is a normal Banach $l^1(S)$-$l^1(E)$-module. Choose $n \in l^\infty(S_G)^*$ with $\langle n, 1\rangle = 1$, and define $D : l^1(S) \to l^\infty(S_G)^*$ by $D(\delta_s) = \chi(\delta_s).\ n - n.\ \chi(\delta_s)$. Moreover, D attains its values in the weak*-closed submodule $(\dfrac{l^\infty(S_G)}{\mathbb{C}})^*$. Since $l^1(S)$ is χ-Connes module amenable, then D is inner. Consequently, there exists $\tilde{n} \in (\dfrac{l^\infty(S_G)}{\mathbb{C}})^*$ such that $D(\delta_s) = ad_{\tilde{n}}$, so

$$\tilde{\chi}(\delta_{\pi(s)}).\ n - n.\ \tilde{\chi}(\delta_{\pi(s)}) = \tilde{\chi}(\delta_{\pi(s)}).\ \tilde{n} - \tilde{n}.\ \tilde{\chi}(\delta_{\pi(s)}).$$

For each $f \in l^\infty(S_G)$,

$$\langle \tilde{\chi}(\delta_{\pi(s)}).\ (n - \tilde{n}) - (n - \tilde{n}).\ \tilde{\chi}(\delta_{\pi(s)}),\ f\rangle = 0.$$

Now put $m := n - \tilde{n} \in l^\infty(S_G)^*$, we have

$$\langle \tilde{\chi}(\delta_{\pi(s)}).\ m - m.\ \tilde{\chi}(\delta_{\pi(s)}),\ f\rangle = 0.$$

By a similar argument as in [18, Lemma 7.1.1], there exists a net $\{f_\alpha\}$ of $l^1(S_G)$ such that $\int f_\alpha = 1$ and $\parallel \tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)}) \parallel \to 0$.

Now let $f \in c_0(S_G \times S_G)$. Take $\epsilon > 0$ and consider a compact set K such that $\|f(x)\|_{S_G \setminus K} < \sqrt{\epsilon}$ and

$$\sup_{s\in K} \| \tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)}) \| < \frac{\sqrt{\epsilon}}{\|f\|}.$$

Since the quotient map is continuous and open, then by [20, Proposition 3.1] we have $\mathcal{L}^2_{w_{k^*}}(l^1(S_G), \mathbb{C}) = c_0(S_G \times S_G)$. Then we may define

$$\langle M,\ f\rangle = \lim_\alpha \int f(\tilde{\chi}(\delta_{\pi(x^*)}), \tilde{\chi}(\delta_{\pi(x)})) f_\alpha(x) dx.$$

By the above argument, for each $s \in S$ there exists α_0 such that for each $\alpha > \alpha_0$, $\| \tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \ \tilde{\chi}(\delta_{\pi(s)})\| < \frac{\sqrt{\epsilon}}{2}$. Hence

$$\begin{aligned}
\langle &\quad \tilde{\chi}(\delta_{\pi(s)}).\ M - M.\ \tilde{\chi}(\delta_{\pi(s)}),\ f\rangle = \langle M,\ f.\ \tilde{\chi}(\delta_{\pi(s)}) - \ \tilde{\chi}(\delta_{\pi(s)}).\ f\rangle \\
= &\quad \lim_\alpha \int \Big(f(\tilde{\chi}(\delta_{\pi(s)\pi(x^*)}), \tilde{\chi}(\delta_{\pi(x)})) - f(\tilde{\chi}(\delta_{\pi(x^*)}), \tilde{\chi}(\delta_{\pi(xs)}))\Big) f_\alpha(x) dx \\
\leq &\quad \|f\|_{S_G\setminus K} \|\tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)})\| \\
+ &\quad |f\|_K \|\tilde{\chi}(\delta_{\pi(s)}) * f_\alpha - f_\alpha * \tilde{\chi}(\delta_{\pi(s)})\| < \epsilon.
\end{aligned}$$

Also for each s

$$\begin{aligned}
\tilde{w}^{**}(M).\tilde{\chi}(\delta_{\pi(s)}) &= \langle M, \tilde{w}^*(\tilde{\chi}(\delta_{\pi(s)}))\rangle \\
&= \lim_\alpha \int (\tilde{w}^*(\tilde{\chi}(\delta_{\pi(s)})))(\tilde{\chi}(\delta_{\pi(x^*)}), \tilde{\chi}(\delta_{\pi(x)})) f_\alpha(x) dx \\
&= \lim_\alpha \int \tilde{\chi}(\delta_{\pi(s)})\tilde{\chi}(\delta_{\pi(x^*)})\tilde{\chi}(\delta_{\pi(x)}) f_\alpha(x) dx \\
&= \lim_\alpha \int \tilde{\chi}(\delta_{\pi(s)}\delta_{\pi(x^*)}\delta_{\pi(x)}) f_\alpha(x) dx \\
&= \lim_\alpha \tilde{\chi}(\delta_{\pi(s)}) \int f_\alpha(x) dx = \tilde{\chi}(\delta_{\pi(s)}).
\end{aligned}$$

Consequently, M is a χ-normal module virtual diagonal for $l^1(S)$. □

Corollary 3.2. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E and let $l^1(S)$ be a Banach $l^1(E)$-module. Then the following are equivalent:*
(i) *$l^1(S)$ is Connes module amenable.*
(ii) *$l^1(S)$ has a module normal virtual diagonal.*

Proof. This follows immediately from Theorem 2.8 and Theorem 3.1. □

Example 10. Let $(\mathbb{N}, \vee)$ be the semigroup of positive integers with maximum operation. Since $\mathbb{N}$ is weakly cancellative, then $l^1(\mathbb{N})$ is a dual Banach algebra with predual $c_0(\mathbb{N})$. By [8, Theorem 5.13], $l^1(\mathbb{N})$

is not Connes amenable. Moreover $l^1(\mathbb{N})$ is module amenable on $l^1(E_{\mathbb{N}})$, so it is Connes module amenable (see [2]).

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φ-CONNES MODULE AMENABILITY OF DUAL BANACH ALGEBRAS

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φ-کنز میانگین‌پذیری مدولی از جبرهای باناخ دوگان

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در این مقاله φ-کنز میانگین‌پذیری مدولی از جبرهای باناخ دوگان A را تعریف می‌کنیم که در آن φ یک همریختی مدولی کراندار از A به A بوده که ضعیف ستاره پیوسته است. بررسی φ مدول نرمال و قطرهای واقعی از اهداف این مقاله است. فرض کنید S یک نیم‌گروه معکوس و حذفی ضعیف بوده و E زیر نیم‌گروه از خودتوان های S باشد. اگر χ یک همریختی مدولی کراندار و ضعیف ستاره پیوسته از $\chi l^{1}(E) l^{1}(S) l^{1}(S)^{1}(S)1$ کنز میانگین‌پذیر مدولی باشد، آن‌گاه دارای یک χ-مدول نرمال و قطر واقعی است. در حالتی که $\chi = id$ عکس این موضوع درست است.

کلمات کلیدی: جبرهای باناخ، میانگین‌پذیری مدولی، اشتقاق و جبر نیم‌گروهی.