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CHARACTERIZATION ON χ-CONNES MODULE AMENABILITY FOR SEMIGROUP ALGEBRAS

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ABSTRACT. We investigate χ-Connes module amenability for semigroup algebras, where χ is a module homomorphism on semigroup algebra that is continuous with respect to ω^*-topology and S is an inverse weakly cancellative semigroup. Also, we study the notion of χ-module normal, virtual diagonals in semigroup algebras. Other hereditary properties in this direction are also obtained.

1. Introduction

In [1], Amini introduced the concept of module amenability for Banach algebras, and proved that when S is an inverse semigroup with subsemigroup E of idempotents, then $l^1(S)$ as a Banach module over $U = l^1(E)$ is module amenable if and only if S is amenable. We may refer the reader e.g. to [1, 4, 5], for more informations. In this paper, we study the concept of χ-Connes module amenability and give a characterization of χ-Connes module amenability in terms of χ-modul normal virtual diagonals.

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* Speaker.

2. Main results

Let S be a semigroup. Then S is named cancellative semigroup, if for every $r, s \neq t \in S$ we have $rs \neq rt$ and $sr \neq tr$.

A discrete semigroup S is called an inverse semigroup if for each $x \in S$ there is a unique element $x^* \in S$ such that $xx^*x = x$ and $x^*xx^* = x^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotent elements of S is denoted by E. For $s \in S$, we define $L_s, R_s : S \to S$ by $L_s(t) = st, R_s(t) = ts; (t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative. If S is a weakly cancellative semigroup, then $l^1(S)$ is a dual Banach algebra with predual $c_0(S)$[2].

Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let I be the closed ideal of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ generated by elements of the form $\alpha.(a \otimes b) - (a \otimes b).\alpha$, for $a, b \in \mathcal{A}$ and $\alpha \in \mathcal{U}$. $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{A}$ is defined to be the quitiont Banach space $\frac{\mathcal{A}\widehat{\otimes}\mathcal{A}}{I}$.

Let J be the closed ideal of $\mathcal{A}$ generated by elements of the form $(\alpha.a).b - a.(b.\alpha)$. In this paper we let that $\mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})$ denote the separately ω^*-continuous two-linear maps from $\frac{\mathcal{A}}{J} \times \frac{\mathcal{A}}{J}$ to $\mathbb{C}$, $\tilde{\omega}^* : \mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{A} \to \frac{\mathcal{A}}{J}$ be the multiplication operator with $\tilde{\omega}(a \otimes b + I) = ab + J$ and $\tilde{\varphi} : \frac{\mathcal{A}}{J} \to \frac{\mathcal{A}}{J}$ be the map that is defined by $\tilde{\varphi}(a + J) = \varphi(a) + J, \ a \in \mathcal{A}$.

Definition 2.1. Let $\mathcal{A}$ be a dual Banach algebra. A module homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is a map $\varphi : \mathcal{A} \to \mathcal{A}$ with

$$\varphi(\alpha.a + b.\beta) = \alpha.\varphi(a) + \varphi(b).\beta, \quad \varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Definition 2.2. Let $\mathcal{A}$ be a dual Banach algebra and $\varphi : \mathcal{A} \to \mathcal{A}$ be a bounded ω^*-continuous module homomorphism. An element $M \in \mathcal{L}^2_{\omega^*}(\frac{\mathcal{A}}{J}, \mathbb{C})^*$ is called a φ-module normal virtual diagonal for $\mathcal{A}$ if $\tilde{\omega}^{**}(M)$ is an identity for $\frac{\varphi(\mathcal{A})}{J}$ and $M.\tilde{\varphi}(c + J) = \tilde{\varphi}(c + J)$ where $c \in \mathcal{A}$.

Let X be a dual Banach $\mathcal{A}$-bimodule. X is called normal if for each $x \in X$, the maps

$$\mathcal{A} \to X; \qquad a \to a.x, \quad a \to x.a$$

are ω^*-continuous. If moreover X is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in X$

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

then X is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly for the right and two sided actions. Also, X is called symmetric, if $\alpha.x = x.\alpha$ for all $\alpha \in \mathcal{U}$ and $x \in X$.

Throughout this paper $\mathcal{H}_{\omega^*}(\mathcal{A})$ will denotes the space of all bounded module homomorphisms from $\mathcal{A}$ to $\mathcal{A}$ that are ω^*-continuous.

Definition 2.3. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\varphi \in \mathcal{H}_{\omega^*}(\mathcal{A})$ and let that X be a dual Banach $\mathcal{A}$-bimodule. A bounded map $D_\mathcal{U} : \mathcal{A} \to X$ is called a module φ-derivation if for every $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathcal{U}$, we have

$$D_\mathcal{U}(\alpha.a \pm b.\beta) = \alpha.D_\mathcal{U}(a) \pm D_\mathcal{U}(b).\beta, \quad D_\mathcal{U}(ab) = D_\mathcal{U}(a).\varphi(b) + \varphi(a).D_\mathcal{U}(b).$$

When X is symmetric, each $x \in X$ defines a module φ-derivation

$$(D_\mathcal{U})_x(a) = \varphi(a).x - x.\varphi(a) \qquad (a \in \mathcal{A}).$$

Derivations of this form are called inner module φ-derivation.

Definition 2.4. Let $\mathcal{A}$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-module and $\varphi \in \mathcal{H}_{\omega^*}(\mathcal{A})$. $\mathcal{A}$ is called φ-Connes module amenable if for any symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module X, each ω^*-continuous module φ-derivation $D_\mathcal{U} : \mathcal{A} \to X$ is inner.

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}$ be a unital dual Banach $\mathcal{U}$-module and let $\mathcal{A}$ has an id-module normal virtual diagonal. Then $\mathcal{A}$ is id-Connes module amenable.*

Proof. Let X be a symmetric normal Banach $\mathcal{A}$-$\mathcal{U}$-module. We first note that $\mathcal{A}$ has an identity. It is therefore sufficient for $\mathcal{A}$ to be id-Connes module amenable that we suppose that X is unital. ☐

Remark 2.6. In Theorem 2.5 it is shown that if a unital Banach algebra $\mathcal{A}$ has an id-module normal virtual diagonal, then $\mathcal{A}$ is id-Connes module amenable. Let S be a semigroup, it would be interesting to know that the converse holds for inverse semigroup algebra $l^1(S)$. Thus for an inverse semigroup S, we consider an equivalence relation on S where $s \sim t$ if and only if there is $e \in E$ such that $se = te$. The quotient semigroup $S_G = \frac{S}{\sim}$ is a group [3]. Also, E is a symmetric subsemigroup of S. Therefore, $l^1(S)$ is a Banach $l^1(E)$-module with compatible canonical actions. Let $l^1(E)$ acts on $l^1(S)$ via

$$\delta_e.\delta_s = \delta_s, \quad \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e \qquad (s \in S, e \in E).$$

The following theorem is the main result of this paper.

Theorem 2.7. *Let S be a weakly cancellative inverse semigroup with idempotents E. let $l^1(S)$ be a Banach $l^1(E)$- module and let $\chi \in \mathcal{H}_{\omega^*}(l^1(S))$. If $l^1(S)$ is χ-Connes module amenable, then there exists a χ-module normal virtual diagonal for $l^1(S)$.*

Proof. Let $\pi : S \to S_G$ be the quotient map. By [1, Lemma 3.2], we define a bimodule action of $l^1(S)$ on $l^\infty(S_G)$ by

$$\delta_s.x = \delta_{\pi(s)} * x, \quad x.\delta_s = x * \delta_{\pi(s)} \qquad (s \in S, x \in l^\infty(S_G)).$$

This completes the proof. □

Theorem 2.8. *Let S be a weakly cancellative semigroup with idempotents E and let $l^1(S)$ be a unital dual Banach $l^1(E)$-module. Moreover, let $l^1(S)\widehat{\otimes}_{l^1(E)} \cdots \widehat{\otimes}_{l^1(E)} l^1(S)$ be a dual Banach $l^1(E)$-module and $\chi \in \mathcal{H}_{\omega^*}(l^1(S))$. Then $\overbrace{l^1(S)\widehat{\otimes}_{l^1(E)} \cdots \widehat{\otimes}_{l^1(E)} l^1(S)}^{n\ times}$ is $\overbrace{\chi \otimes_{l^1(E)} \cdots \otimes_{l^1(E)} \chi}^{n\ times}$-Connes module amenable if and only if $l^1(S)$ is χ-Connes module amenable.*

Example 2.9. Let $(\mathbb{N}, \vee)$ be the semigroup of positive integers with maximum operation. Since $\mathbb{N}$ is weakly cancellative, then $l^1(\mathbb{N})$ is a dual Banach algebra with predual $c_0(\mathbb{N})$. By [2, Theorem 5.13], $l^1(\mathbb{N})$ is not Connes amenable. So, it is Connes module amenable.

3. Conclusion

In this paper we show that if S is a weakly cancellative inverse semigroup with idempotents E and χ is a bounded module homomorphism from $l^1(S)$ to itself, then $l^1(S)$ has χ-module normal virtual diagonal.

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