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On φ-Connes Module Amenability of Dual Banach Algebras and φ-splitting

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Abstract. In this paper, we define φ-Connes module amenability of a dual Banach algebra $\mathcal{A}$, where φ is a ω^*-continuous bounded module homomorphism from $\mathcal{A}$ onto itself. We obtain the relation between φ-Connes module amenability of $\mathcal{A}$ and φ-splitting of the certain short exact sequence. We show that if S is a weakly cancellative inverse semigroup with subsemigroup E_S of idempotents and $l^1(S)$ as a Banach module over $l^1(E_S)$ is χ-Connes module amenable, then the short exact sequence is χ-splitting that χ is a ω^*-continuous bounded module homomorphism from $l^1(S)$ onto itself. Other results in this direction are also obtained.

Keywords: φ-σwc virtual diagonal, φ-Connes module amenability, φ-splitting, short exact sequence, semigroup algebra, weakly cancellative semigroup

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1. Introduction

Ghaffari and Javadi in [4], investigated φ-Connes amenability for dual Banach algebras and semigroup algebras, where φ was an homomorphism from a Banach algebra on $\mathbb{C}$. Also, the $\chi \otimes \eta$-strong Connes amenability of certain dual Banach algebras is investigated by Tamimi and Ghaffari in [11]. Also in [5], Ghaffari et al. investigated φ-Connes module amenability of dual Banach algebras that φ is a ω^*-continuous bounded module homomorphism from a Banach algebra on itself. χ-module Connes amenability of semigroup algebras is studied by the authors in [10]. What is the relation between φ-splitting and φ-Connes module amenability, where φ is ω^*-continuous homomorphism from Banach algebra to itself ? Motivated by above question and [9], to study φ-Connes amenability and φ-splitting. We recall that for Banach algebra $\mathcal{A}$, the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule in the canonical way. Now, we define the map $\mathcal{A}$-bimodule homomorphism $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ by $\pi(a \otimes b) = ab$. A Banach $\mathcal{A}$-bimodule E is dual if there is a closed submodule $E_* \subseteq E^*$, predual of E, such that $E = (E_*)^*$. A dual Banach $\mathcal{A}$-bimodule E is normal if the module actions of $\mathcal{A}$ on E are ω^*-continuous. A Banach algebra is dual if it is dual as a Banach $\mathcal{A}$-bimodule. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a Banach $\mathcal{A}$-bimodule. Then $\sigma wc(E)$, a closed submodule of E,

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stands for the set of all elements $x \in E$ such that the following maps are ω^*-ω continuous

$$\mathcal{A} \longrightarrow E; \quad a \longmapsto a.x, \ \ a \longmapsto x.a.$$

The Banach $\mathcal{A}$-bimodules E that are relevant to us are those the left action is of the form $a.x = \varphi(a)x$. For the brevity's sake, such E will occasionally be called a Banach φ-bimodule.
Throughout the paper, $\Delta(\mathcal{A})$ and $\Delta_{\omega^*}(\mathcal{A})$ will denote the sets of all homomorphisms and ω^*-continuous homomorphisms from the Banach algebra $\mathcal{A}$ onto $\mathbb{C}$, respectively. Ghaffari and Javadi in [4], investigated ϕ-Connes amenability for dual Banach algebras, where ϕ is an homomorphism from a Banach algebra on $\mathbb{C}$. Also, several characterizations of $\widehat{\chi}$-Connes amenability of semigroup algebras were introduced by these two authors, where χ is a nonzero bounded continuous character on unital weakly cnacellative semigroup S and the map $\widehat{\chi}$ is defined on semigroup algebra $l^1(S)$. Weak module amenability for semigroup algebras is studied by Amini and Ebrahimi bagha in [1]. Recently, in [5], Ghaffari et al. investigated φ-Connes module amenability of dual Banach algebras that ψ is a ω^*-continuous bounded module homomorphism from a Banach algebra on itself. In [2], the concept of module amenability for Banach algebras is introduced. Also, it is proved that when S is an inverse semigroup with subsemigroup E_S of idempotents, then $l^1(S)$ as a Banach module over $\mathcal{U} = l^1(E_S)$ is module amenable if and only if S is amenable. For more information and details of module amenability, we may refer the reader to [2, 10]. Ghaffari et al. studied dual of group algebras under a locally convex topology [7]. In fact, we give a characterization of φ-Connes module amenability of a dual Banach algebra in terms of so-called φ-splitting of the certain short exact sequences (Theorem 3.3). Also, the mentioned concepts and details are shown for semigroup algebras in Theorem 4.4. In Theorem 3.4, by letting that $\mathcal{A}$ and $\mathcal{B}$ are φ and ψ-Connes module amenable Banach algebras respectively, that both of $\varphi : \mathcal{A} \to \mathcal{A}$ and $\psi : \mathcal{B} \to \mathcal{B}$, are ω^*-continuous bounded module homomorphisms, we show that this property is transferred from $\mathcal{A}$ and $\mathcal{B}$ to the special tensor product of their. In finally, it is presented a corollary and an example in this direction.

2. Preliminary Notations

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach A-bimodule. A derivation from $\mathcal{A}$ to E is a bounded, linear map $D : \mathcal{A} \to E$ satisfying $D(ab) = a.D(b) + D(a).b$ $(a, b \in \mathcal{A})$. A derivation $D : \mathcal{A} \to E$ is called inner if there is $x \in E$ such that $Da = a.x - x.a$ $(a \in \mathcal{A})$.

Definition 2.1. Let $\mathcal{A}$ be a Banach algebra, and let $3 \leq n \in \mathbb{N}$. A sequence

$$\mathcal{A}_1 \overset{\varphi_1}{\to} \mathcal{A}_2 \overset{\varphi_2}{\to} .... \overset{\varphi_{n-1}}{\longrightarrow} \mathcal{A}_n$$

of $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i \in \{2, ..., n-1\}$ is called exact at position $i = 2, ..., n-1$ if $\varphi_{i-1} = ker\varphi_i$. It is called exact if it is exact at every position $i \in \{2, ..., n-1\}$.

We restrict ourselves to exact sequences with few bimodules, and a few bimodules (short exact sequences) respectively. Therefore, an exact sequence of the following form

$$0 \to \mathcal{A}_1 \overset{\varphi}{\to} \mathcal{A}_2 \overset{\psi}{\to} \mathcal{A}_3 \to 0$$

is called a short exact sequence.

In the following we define the admissible and splitting short exact sequences.

Definition 2.2. Let $\mathcal{A}$ be a Banach algebra. A short exact sequence

$$\Theta : 0 \to \mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_{n-1}} \mathcal{A}_n \to 0$$

of Banach $\mathcal{A}$-bimodules $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n$ and $\mathcal{A}$-bimodule homomorphisms $\varphi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ for $i = 1, 2, ...n-1$ is admissible, if there exists a bounded linear map $\rho_i : \mathcal{A}_{i+1} \to \mathcal{A}_i$ such that $\rho_i o \varphi_i$ on $\mathcal{A}_i$ for $i = 1, 2, ...n-1$ is the identity map. Further, Θ splits if we may choose ρ_i to be an $\mathcal{A}$-bimodule homomorphism.

Definition 2.3. Let $\mathcal{A} = (\mathcal{A}_*)^*$ be an unital dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$. We say that $\sum \varphi$-splits if there exists a bounded linear map $\rho : \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*) \to \mathcal{A}_*$ such that $\rho o \pi^*(\varphi) = \varphi$ and $\rho(T.a) = \varphi(a)\rho(T)$, for all $a \in \mathcal{A}$ and $T \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$.

Definition 2.4. Let $\mathcal{A}$ be a dual Banach algebra, and let $\varphi \in \Delta_{\omega^*}(\mathcal{A}) \cap \mathcal{A}_*$. An element $M \in \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ is a φ-σwc virtual diagonal for $\mathcal{A}$ if
(i) $a.M = \varphi(a)M, \quad (a \in \mathcal{A})$;
(ii) $\langle \varphi \otimes \varphi, M \rangle = 1$.

In throughout this paper, let $\otimes_\omega$ stand for the injective tensor product of Banach algebras.
We consider the following short exact sequences, which have three non-zero terms:

$$\sum_{\varphi} : 0 \to \mathcal{A}_* \xrightarrow{\pi^*_{\mathcal{A}}} \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \sigma wc(\mathcal{A}\widehat{\otimes}\mathcal{A})^*/\pi^*_{\mathcal{A}}(\mathcal{A}_*) \to 0,$$

$$\sum_{\psi} : 0 \to \mathcal{B}_* \xrightarrow{\pi^*_{\mathcal{B}}} \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^* \to \sigma wc(\mathcal{B}\widehat{\otimes}\mathcal{B})^*/\pi^*_{\mathcal{B}}(\mathcal{B}_*) \to 0$$

and

$$\sum_{\varphi\otimes\psi} : 0 \to \mathcal{A}_*\otimes_\omega\mathcal{B}_* \xrightarrow{\pi^*_{\mathcal{A}\widehat{\otimes}\mathcal{B}}} \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{B})\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{B}))^* \to \sigma wc((\mathcal{A}\widehat{\otimes}\mathcal{B})\widehat{\otimes}(\mathcal{A}\widehat{\otimes}\mathcal{B}))^*/\pi^*_{\mathcal{A}\widehat{\otimes}\mathcal{B}}(\mathcal{A}_*\otimes_\omega\mathcal{B}_*) \to 0.$$

Definition 2.5. ([4, Definition 2.1]) Let $\mathcal{A}$ be a dual Banach algebra and $\varphi \in \Delta(\mathcal{A}) \cap \mathcal{A}_*$. $\mathcal{A}$ is φ-Connes amenable if for every normal φ-bimodule E, every bounded ω^*-continuous derivation $D : \mathcal{A} \to E$ is inner.

3. φ-Connes module amenability and φ-splitting

In this section our main aim is investigation of the relation between notions of φ-Connes module amenability of dual Banach algebras and φ-splitting of the short exact sequences. Also, we discuss some hereditary properties of φ-Connes module amenability. The following definitions are analogue to [5, 10].
Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, and $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}$ is a Banach $\mathcal{U}$-bimodule via,

$$\alpha.(ab) = (\alpha.a).b, \quad (\alpha\beta).a = \alpha.(\beta.a) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathcal{U}).$$

Let E be a dual Banach $\mathcal{A}$-bimodule. E is called normal if for each $x \in E$, the maps

$$\mathcal{A} \to E; \qquad a \to a.x, \quad a \to x.a$$

are ω^*- continuous. If moreover E is a $\mathcal{U}$-bimodule such that for $a \in \mathcal{A}, \alpha \in \mathcal{U}$ and $x \in E$

$$\alpha.(a.x) = (\alpha.a).x, \quad (a.\alpha).x = a.(\alpha.x), \quad (\alpha.x).a = \alpha.(x.a),$$

then E is called a normal Banach left $\mathcal{A}$-$\mathcal{U}$-module. Similarly for the right and two sided actions. Also, E is called commutative, if

$$\alpha.x = x.\alpha \qquad (\alpha \in \mathcal{U}, x \in E).$$

A module homomorphism from $\mathcal{A}_*$ to $\mathcal{A}_*$ is a map $\varphi : \mathcal{A}_* \to \mathcal{A}_*$ with

$$\varphi(\alpha.a + b.\beta) = \alpha.\varphi(a) + \varphi(b).\beta, \quad \varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathcal{A}_*, \alpha, \beta \in \mathcal{U}).$$

It is obvious that multiplication in $\mathcal{A}$ is ω^*-continuous. Consider $\mathcal{A}_*$ as dual $\mathcal{A}_*$-module with predual $\mathcal{A}_*$. So, we shall suppose that $\mathcal{A}_*$ takes ω^*-topology. $\mathcal{HOM}^b_{\omega^*}(\mathcal{A}_*)$ will denotes the space of all bounded module homomorphisms from $\mathcal{A}_*$ to $\mathcal{A}_*$ that are ω^*-continuous.

Now, in the following we present some definitions.

Definition 3.1. ([5], P. 71) Let $\mathcal{A} = (\mathcal{A}_*)^*$ be a dual Banach algebra, $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A}_*)$. let E be a dual Banach $\mathcal{A}_*$-bimodule. A bounded map $D_{\mathcal{U}} : \mathcal{A}_* \to E$ is called a module φ-derivation if

$$D_{\mathcal{U}}(\alpha.a \pm b.\beta) = \alpha.D_{\mathcal{U}}(a) \pm D_{\mathcal{U}}(b).\beta, \quad D_{\mathcal{U}}(ab) = D_{\mathcal{U}}(a).\varphi(b) + \varphi(a).D_{\mathcal{U}}(b), \quad (a, b \in \mathcal{A}_*, \alpha, \beta \in \mathcal{U}).$$

When E is commutative, each $x \in E$ defines a module φ-derivation

$$(D_{\mathcal{U}})_x(a) = \varphi(a).x - x.\varphi(a) \qquad (a \in \mathcal{A}_*).$$

Derivations of this form are called inner module φ-derivation.

Definition 3.2. ([5], P. 71) Let $\mathcal{A}_*$ be a dual Banach algebra, $\mathcal{U}$ be a Banach algebra such that $\mathcal{A}_*$ is a Banach $\mathcal{U}$-module and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A}_*)$. $\mathcal{A}_*$ is called φ-Connes module amenable if for any commutative normal Banach $\mathcal{A}_*$-$\mathcal{U}$-module E, each ω^*-continuous module φ-derivation $D_{\mathcal{U}} : \mathcal{A}_* \to E$ is inner.

Recall that if φ is identity map on $\mathcal{A}$, then id-Connes module amenability is called Connes module amenability. Also, by the proof of [2, Proposition 2.1], Connes amenability of $\mathcal{A}$ implies its Connes module amenability in the case where $\mathcal{U}$ has a bounded approximate identity for $\mathcal{A}$. In continuation, example 4.6 shows that the converse is false.

Theorem 3.3. *Let $\mathcal{A}_*$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A}_*)$. Then $\mathcal{A}_*$ is φ-Connes module amenable if and only if the short exact sequence Σ_φ φ-splits.*

Proof. Let $\mathcal{A}_*$ be φ-Connes module amenable and E be a commutative normal Banach $\mathcal{A}_*$-$\mathcal{U}$-module and $D_{\mathcal{U}} : \mathcal{A}_* \to E$ be an inner ω^*-continuous module φ-derivation. Without loss of generality, suppose that $\mathcal{A}_*$ is unital and $\varphi(e_{\mathcal{A}_*}) = e_{\mathcal{A}_*}$. This completes the proof.
For the converse suppose that $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A}_*)$ and the short exact sequence Σ_φ φ-splits. $\mathcal{A}$ is φ-Connes amenable. Thertefore $\mathcal{A}$ is φ-Connes module amenable. □

Suppose that $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras such that $\mathcal{A}$ and $\mathcal{B}$ be dual Banach $\mathcal{U}$-modules and $\mathcal{A}\widehat{\otimes}\mathcal{B}$ denotes the projective tensor product of $\mathcal{A}$ and $\mathcal{B}$. Let I be the closed ideal of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ generated by elements of the form $\alpha.(a \otimes b) - (a \otimes b).\alpha$ for $a \in \mathcal{A}, b \in \mathcal{B}$ and $\alpha \in \mathcal{U}$. $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ is defined to be the quitiont Banach space $\frac{\mathcal{A}\widehat{\otimes}\mathcal{B}}{I}$, that is, $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B} \cong \frac{\mathcal{A}\widehat{\otimes}\mathcal{B}}{I}$ [8].

Let $\mathcal{A}, \mathcal{B}$ be commutative Banach $\mathcal{U}$-bimodules and let $\varphi \in \mathcal{HOM}_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. Consider $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ with the product specified by $(a \otimes b)(c \otimes d) = ac \otimes bd$ $(a, c \in \mathcal{A},\ b, d \in \mathcal{B})$. Let $\varphi \otimes \psi$ denotes the elements of $\mathcal{HOM}^b_{\omega^*}(\mathcal{A}\widehat{\otimes}\mathcal{B})$ satisfying $\varphi \otimes \psi(a \otimes b) = \varphi(a) \otimes \psi(b)$ for all $a \in \mathcal{A}, b \in \mathcal{B}$. $\varphi \otimes \psi$ induces a map $\varphi \otimes_{\mathcal{U}} \psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B})$ with $\varphi \otimes_{\mathcal{U}} \psi(a \otimes b) =$

$\varphi(a) \otimes \psi(b) + I$ [3].
By above details, we obtain the following theorem.

THEOREM 3.4. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}, \mathcal{B}$ be unital dual Banach $\mathcal{U}$- modules and let $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. If $\mathcal{A}, \mathcal{B}$ are φ, ψ-Connes module amenable respectively, then $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ is $\varphi\widehat{\otimes}_{\mathcal{U}}\psi$-Connes module amenable.*

THEOREM 3.5. *Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{U}$ be dual Banach algebras, let $\mathcal{A}, \mathcal{B}$ be unital dual Banach $\mathcal{U}$- modules and let $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ be a dual Banach algebra and $\varphi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{A})$, $\psi \in \mathcal{HOM}^b_{\omega^*}(\mathcal{B})$. $\mathcal{A}\widehat{\otimes}_{\mathcal{U}}\mathcal{B}$ is $\varphi\widehat{\otimes}_{\mathcal{U}}\psi$-Connes module amenable if and only if the short exact sequences $\Sigma_{\varphi\widehat{\otimes}_{\mathcal{U}}\psi}$ $\varphi\widehat{\otimes}_{\mathcal{U}}\psi$-splits.*

4. Application

In this section by considering module homomorphisms on semigroup algebras, we investigate φ-splitting for notions of Connes module amenability and the short exact sequences. The following definitions are analogue to [5, 10].

A discrete semigroup S is called an inverse semigroup if for each $t \in S$ there is a unique element $t^* \in S$ such that $tt^*t = t$ and $t^*tt^* = t^*$. The set of idempotent elements of S is denoted by $E_S = \{e \in S;\ \ e = e^* = e^2\}$.

REMARK 4.1. Let S be an inverse semigroup. For $s \in S$, we define $L_s, R_s : S \to S$ by $L_s(t) = st$, $R_s(t) = ts$, $(t \in S)$. If for each $s \in S$, L_s and R_s are finite-to-one maps, then we say that S is weakly cancellative. We know that if S is a weakly cancellative semigroup, then $(c_0(S))^* = l^1(S)$.

DEFINITION 4.2. Let S be a weakly cancellative semigroup, S be an inverse semigroup with idempotents E_S. Let $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$ and $l^1(S)$ be a Banach $l^1(E_S)$-module. An element $M \in \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*)^*$ is a $\chi - \sigma wc$- virtual diagonal for $l^1(S)$ if

$$\delta_s.M = \chi(\delta_s)M, \quad \langle \chi \otimes \chi, M\rangle = 1, \qquad (\delta_s \in l^1(S)).$$

Let $l^1(S) = (l^1(S)_*)^*$ be an unital dual Banach algebra. Then we consider the following short exact sequence of $l^1(S)$ -bimodules,

$$\sum_{\chi} : 0 \longrightarrow l^1(S)_* \xrightarrow{\pi^*_\chi} \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*) \longrightarrow \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*)/\pi^*_\chi(l^1(S)_*) \longrightarrow 0$$

Now, we present an important definition.

DEFINITION 4.3. Let S be a weakly cancellative inverse semigroup. Let $l^1(S) = (c_0(S))^*$ be an unital dual Banach algebra, and let $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$. We say that $\sum_\chi$ χ-splits if there exists a bounded linear map $\rho : \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*) \to l^1(S)_* = c_0(S)$ such that $\rho o\pi^*_\chi(\chi) = \chi$ and $\rho(T.\delta_s) = \chi(\delta_s)\rho(T)$, for all $\delta_s \in l^1(S)$, $T \in \sigma wc((l^1(S)\widehat{\otimes}l^1(S))^*)$ and $\pi^*_\chi : l^1(S) \otimes l^1(S) \to l^1(S)$.

THEOREM 4.4. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E_S, $\chi \in \mathcal{HOM}^b_{\omega^*}(l^1(S))$ and let $l^1(S)$ be a Banach $l^1(E_S)$-module. Then $l^1(S)$ is χ-Connes module amenable if and only if the short exact sequences Σ_χ χ-splits.*

COROLLARY 4.5. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E_S and let $l^1(S)$ be a Banach $l^1(E_S)$-module. Then $l^1(S)$ is Connes module amenable if and only if the short exact sequences $\Sigma_{\chi=id}$ splits.*

Example 4.6. Let $(\mathbb{N}; \vee : \mathbb{N} \to \mathbb{N})$ be the semigroup of natural numbers with maximum operation. We know that $\mathbb{N}$ is weakly cancellative, because

$$L_s : \mathbb{N} \to \mathbb{N},\ \ L_s(n) = sn\ \ and\ \ R_s : \mathbb{N} \to \mathbb{N},\ \ R_s(n) = ns;\ \ (n \in \mathbb{N}),$$

are not one to one. Then $l^1(\mathbb{N})$ is a dual Banach algebra that $(c_0(\mathbb{N}))^* = l^1(\mathbb{N})$. $l^1(\mathbb{N})$ is not Connes amenable. Moreover, $l^1(\mathbb{N})$ is module amenable on $l^1(E_{\mathbb{N}})$, so it is Connes module amenable. Suppose that M is a $\chi - \sigma wc$- virtual diagonal for $l^1(\mathbb{N})$. Now if we define $\rho : \sigma wc((l^1(\mathbb{N})\widehat{\otimes}l^1(\mathbb{N}))^*) \to l^1(\mathbb{N})_*$ by

$$\langle \delta_n, \rho(T)\rangle = \langle T.\delta_n, M\rangle, \qquad \big(n \in \mathbb{N}, \delta_n \in l^1(\mathbb{N}), T \in \sigma wc((l^1(\mathbb{N})\widehat{\otimes}l^1(\mathbb{N}))^*)\big)$$

We obtain

$$\langle \delta_n, \rho o \pi_\chi^*(\chi)\rangle = \langle \pi_\chi^*(\chi).\delta_n, M\rangle = \langle \pi_\chi^*(\chi), \delta_n.M\rangle = \chi(\delta_n)\langle \pi_\chi^*(\chi), M\rangle = \chi(\delta_n).$$

Next for $m, n \in \mathbb{N}, \delta_n, \delta_m \in l^1(\mathbb{N})$ we have

$$\begin{aligned}\langle \delta_m, \rho(T.\delta_n)\rangle &= \langle T.\delta_n\delta_m, M\rangle = \langle T, \delta_n\delta_m.M\rangle = \chi(\delta_n\delta_m)\langle T, M\rangle \\ &= \chi(\delta_n)\langle T, \delta_m.M\rangle = \chi(\delta_n)\langle T.\delta_m, M\rangle = \chi(\delta_n)\langle \delta_m, \rho(T)\rangle.\end{aligned}$$

All in all, the short exact sequences $\Sigma_{\chi=id}$ splits.

Corollary 4.7. *Let S be a weakly cancellative semigroup, let S be an inverse semigroup with idempotents E_S and let $l^1(S)$ be a Banach $l^1(E_S)$-module. Then $l^1(S)\otimes l^1(S)$ is $\chi\otimes\eta$ Connes module amenable if and only if the short exact sequences $\Sigma_{\chi\otimes\eta}$ $\chi\otimes\eta$-splits.*

5. Conclusion

In this paper, we studied the relation between φ-splitting and φ-Connes module amenability, where φ is a ω^*-continuous bounded module homomorphism from Banach algebra $\mathcal{A}$ onto $\mathcal{A}$. Also, by considering that S is a weakly cancellative semigroup, then we obtain similar results for semigroup algebras $l^1(S)$ and $l^1(S)\otimes l^1(S)$.

References

1. M. Amini and D. Ebrahimi Bagha, *Weak module amenability for semigroup algebras*, Semigroup Forum, 71 (2005), 18-26.
2. M. Amini, *Module amenability for semigroup algebras*, Semigroup Forum, 69 (2004), 243–254.
3. H. G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monogr. ser, Clarendon Press, 2000.
4. A. Ghaffari and S. Javadi, *φ-Connes amenability of dual Banach algebras*, Bull. Iranian Math. Soc. 43 (2017), 25–39.
5. A. Ghaffari, S. Javadi and E. Tamimi, *φ-Connes module amenability of dual Banach algebras*, Journal of Algebraic Systems, 8(1) (2020), 69-82.
6. A. Ghaffari, S. Javadi and E. Tamimi, *Connes Amenability for Certain Product of Banach Algebras*, Wavelets and Linear Algebra, 9(1) (2022), 1-14.
7. A. Ghaffari, M. Sheibani, E Tamimi, *On dual of group algebras under a locally convex topology*, Journal of Advanced Mathematical Modeling (JAMM), 13 (2023), no.2 1-9.
8. M. A. Rieffel, *Induced Banach representations of Banach algebras and locally compact groups*, J. Func. Analysis, 1 (1967), 443–491.
9. V. Runde, *Lectures on Amenability*, Lecture Notes in Mathematics 1774, Springer Verlag, Berlin, 2002.
10. E. Tamimi and A. Ghaffari, *A Survey on χ-module Connes amenability of semigroup algebras*, Journal of Algebraic Systems, 29 (2023), Accepted to Online Publish. https://doi.org/10.21203/rs.3.rs-2676837/v1
11. E. Tamimi and A. Ghaffari, *On $\chi\otimes\eta$-strong Connes amenability of certain dual Banach algebras*, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math., 31 (2024), no. 1, Accepted, https://doi.org/10.7468/jksmeb.2024.31.1.**